Technology Adoption and Human Capital Accumulation:

An Exactly Solved Framework for Unifying the Empirics on Automation and Education*

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Abstract

Labor market skills can depreciate in usefulness and relevance very suddenly, for instance through introduction of new technologies or off-shoring of production – a process referred to as human capital obsolescence. Recently developed task-based frameworks have proven adept at capturing such labor-displacing dynamics alongside the canonical factor-augmenting changes in technology. Nevertheless, none of these models incorporate decisions on human capital investment. Hence, the labor immiseration scenarios predicted so far run contrary to the empirics on the effects of automation. In this study, I remedy this by including considerations of human capital investment based on the conceptual framework of the psychometric literature on skill formation. Moreover, focus of the study is shifted towards institutional adoption of new technologies, rather than their arrival rate. In doing so, the model produces a labor immiseration scenario consistent with around a dozen pivotal empirical findings on returns to education and consequences of automation while solving several puzzles in the literature. Some of these empirics include routine-biased technological change, job market polarization, spells of jobless recovery following automation with simultaneous boosts to productivity, falling labor share of global income, supply-side secular stagnation, convex pattern of skill-premium, diminishing marginal returns to schooling and importance of human capital endowment for adoption of technology. The main mechanism for many of these empirics is shown to be the faster accumulation rate of physical vis-à-vis human capital. Other contributions of this study are providing a richer alternative to the Roy model of occupational choice and earnings inequality in the empirical labor literature and a unification of the task-based framework with the third generation of distributional models in macroeconomics fused with decisions on human capital investment.

Key words: Technology adoption; automation; human capital investment, obsolescence and accumulation; task-based framework; skill diversification; distributional effects; returns to education; intergenerational and occupational mobility

JEL Codes: D63, E22, E23, E24, I26, J24, J62, O33, O41

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1 Introduction

Recent literature have reintroduced technological change as a major contributor to labor-displacement with varying impact across performers of different tasks (Acemoglu and Restrepo, 2018*e*; Autor and Salomons, 2018). Indeed, labor market skills can depreciate in usefulness and relevance very suddenly, for instance through introduction of new technologies (automatization and digitalization) or off-shoring of production (Goos, Manning and Salomons, 2014), a process referred to as *human capital obsolescence*. Additionally, the rise of artificial intelligence has given salience to the fear that automating tasks will readily become easier, making the dominant share of technological advancements labor-displacing rather than factor-augmenting.

Recent studies, for instance, find evidence which lends credibility to these misgivings concerning the introduction of industrial robots. Some find that the adoption of industrial robots indeed has led to a decline in employment and wage rates in the United States compared to adjacent commuting zones (Acemoglu and Restrepo, 2017*a*). Other researchers find that automation has caused jobless recoveries in United States (Graetz and Michaels, 2017) as opposed to other developed countries. This is while the US has been shown to be better at implementing new technology due to uncompromising management practices viewed in terms of growth in total factor productivity (Bloom, Sadun and Van Reenen, 2012). Moreover, the labor share of the value added in production has decreased across the board especially among performers of varying routine tasks (Acemoglu and Restrepo, 2017*a*; Autor and Salomons, 2018; Graetz and Michaels, 2018).

Indeed, these findings are in line with the emerging literature on routine-biased technological change[1] – henceforth RBTC – which document simultaneous job-market polarization and increased productivity. Such results gain additional credence once compared to the historical slumps in levels of wage and employment alongside productivity-increasing technological changes (Allen, 2009; Ford, 2015). Moreover, a stylized fact has been noted by researchers that recent investments in automation has failed to deliver on the expected level of increase in productivity (Acemoglu and Restrepo, 2017*a*,*b*, 2018*e*) which may be due to mismatch between demand and supply of skills (Acemoglu and Restrepo, 2018*a*). Reports of such mismatch are indeed consistent and common place (cf. e.g. Brynjolfsson and McAfee, 2014).

Previously, however, such unbalanced growth trajectories were either ruled out (cf. e.g. Kaldor, 1957, 1962; Denison, 1974; Barro and Sala-i Martin, 1990; Homer and Sylla, 1996) or deemed transitory as labor shifted from one sector – experiencing automation – to another with more labor-intensive production (Kuznets, 1957, 1973; Chenery, 1960; Kongsamut, Rebelo and Xie, 2001; Ngai and Pissarides, 2007; Acemoglu and Guerrieri, 2008). An example of the latter development is the labor shift away from agriculture and towards manufacturing during early and mid twentieth century (Allen, 2009; Goldin and Katz, 2009). Recent empirical findings, however, undermine several widely popular stylized facts such as constant interest rates and a constant labor share of production (Elsby, Hobijn and Şahin, 2013; OECD, 2015; Piketty, 2014; Eggerts-

[1] See Autor, Levy and Murnane (2003); Goos, Manning and Salomons (2009, 2014); Acemoglu and Autor (2011); Adermon and Gustavsson (2015); Cortes, Jaimovich and Siu (2017); Hershbein and Kahn (2018).

son, Robbins and Wold, 2018; Schwellnus et al., 2018) – which are supported by both of these growth narratives.

As a result, a marginalized and more pessimistic growth framework has gained recent attention. This narrative was first put forth in the seminal works of Marx (1889) – predicting complete eventual labor immiseration – and Baumol (1967) – anticipating total obsolescence of sectors undergoing automatization. Several authors have recently built on these insights in order to match – with varying degrees of alarmism – the empirical evidence on RBTC, job market polarization and the dissipation of middling tasks in the recent two decades.[2] For instance Berg, Buffie and Zanna (2018) find that in most conceivable cases automation has strongly positive and negative effects on growth and equality respectively – even in the optimistic case of robots only being used for a subset tasks or immunity of certain sectors to such automation technology. A more pessimistic view, Susskind (2017) predicts full immiseration of labor through a process of *task encroachment* – as 'advanced capital' accumulates the subset of tasks performed by labor diminishes over time and approaches zero. Feng and Graetz (2019) find evidence of such encroachment by studying job training requirements. They illustrate that employment have been polarized by initial occupational training requirements and shifted towards more complex occupations; and that the relationship between complexity and employment growth is weakest among occupations with low training requirements.

Nevertheless, the most comprehensive theoretical work on capturing the insights of the task-based framework to date most likely is Acemoglu and Restrepo (2018*e*), plus the extension in Acemoglu and Restrepo (2018*c*). The task-based approach offers a more radical form of creative destruction than that of the early Schumpeterian growth framework (cf. e.g. Aghion, 2002; Aghion, Akcigit and Howitt, 2014). In the latter approach the destructive part of technological change is limited to less productive intermediate firms being replaced by ones which are more so or better at patenting their innovations. In the task-based framework, on the other hand, two kinds of destruction patterns are present: labor-skill obsolescence (as a result of automation) and automated tasks becoming obsolete (following new and ground-breaking technology), which both are potentially more ruinous than a number of firms going bust. Acemoglu and Restrepo (2018*e*) illustrate the conditions under which full automation is a feasible growth path: (a) highly patient agents,[3] (b) low interest rates – or equivalently too intensive capital deepening, or (c) a combination of high aggregate productivity, low depreciation rate of capital, high elasticity of intertemporal substitution and low economic growth. Yet, they put aside these scenarios as less interesting, striking a more optimistic tone. A similar approach is made in Acemoglu and Restrepo (2018*c*) although for the distinguished low- and high-skill automating capital stocks.

In this study, I illustrate that these dismissed situations may not in fact be uninteresting theoretical constructs, rather exceedingly feasible development paths given our evidence on human

[2]See Autor, Katz and Kearney (2006); Goos and Manning (2007); Goos, Manning and Salomons (2009); Acemoglu and Autor (2011); Autor (2013, 2015, 2019); Autor and Dorn (2013); Foote and Ryan (2015); Harrigan, Reshef and Toubal (2016); Autor and Salomons (2018).

[3]That is, agents with very low discounting factor.

capital accumulation. In order to do so, I build on the concepts of task-based framework in Zeira (1998), Autor, Levy and Murnane (2003) and Acemoglu and Restrepo (2018*a*,*b*,*c*,*d*,*e*) and augment it with considerations on agents' human capital investments under risk for skill obsolescence. I microfound the model on the psychometric literature regarding skill formation.[4] Consequently, the predictions of this model qualitatively match the empirical literature on returns to education as well, while providing a fiscal incentive structure – rather than the canonical, yet widely criticized, hedonic models of Mincer (1958, 1974).[5] Furthermore, by I show that by viewing labor as human-capital-augmented, the pivotal channel in determining the technological path of economy, is not mainly the *arrival rate* of technology – as suggested by Acemoglu and Restrepo (2018*e*) – but rather the conditions for their *institutional adoption*.

Another distinct feature of this study is the inclusion of uninsurable idiosyncratic wealth shocks, making the model a heterogeneous agent approach rather than a representative one. Hence, I can derive decision rules of the agents as a function of both the individual and aggregate variables in the economy. This is achieved through employing the techniques developed by the third generation of distributional macromodels (cf. e.g. Lucas Jr and Moll, 2014; Ahn et al., 2018; Kaplan, Moll and Violante, 2018; Nuño and Moll, 2018). Thus, I am able to analyze the varying distributional effects of technological shifts on a workforce heterogeneous in both human capital endowment (*ex-ante*) and wealth (*ex-post*). Furthermore, I follow Itskhoki and Moll (2019) in providing an intergenerational interpretation of the model which then predicts dynastic human capital as documented by Long and Ferrie (2007, 2013), Lindahl et al. (2014, 2015) and Turner et al. (2018). Finally, this model provides a richer alternative to the seminal work of Roy (1951) on occupational choice and earnings inequality in the empirical labor literature.

Hence, this project produces an exactly-solved task-based framework in a heterogeneous distributional setting which unifies several (at-times seemingly contradictory) empirical findings in the literature concerning automation and its consequences while being qualitatively consistent with the empirical evidence on skill formation and education acquisition. Below is a list of these empirical evidence pieces and puzzling facts:

I. Evidence from the Psychometric Literature on Skill Formation

1. Self-productivity of human capital: Skills produced at one stage augment the ones attained at later stages (Cunha et al., 2006; Cunha and Heckman, 2007, 2008; Almond and Currie, 2011; Carneiro, Heckman and Vytlacil, 2011; Graff Zivin and Neidell, 2013)
2. Dynamic complementarity: Skills produced at one stage raise the productivity of human capital investment at subsequent stages (Cunha et al., 2006; Cunha and Heckman, 2007, 2008; Cunha, Heckman and Schennach, 2010; Almond and Currie, 2011; Graff Zivin and Neidell, 2013)

[4] See Cunha et al. (2006); Cunha and Heckman (2007, 2008); Almond and Currie (2011); Cunha, Heckman and Schennach (2010); Helmers and Patnam (2011); Graff Zivin and Neidell (2013).

[5] For criticisms and rejections of the Mincerian models see e.g. Heckman, Lochner and Todd (2006, 2008); Bhuller, Mogstad and Salvanes (2017).

3. (Local) cross-productivity of human capital: Stock in one skill eases acquisition of other (related) skills, and vice-versa (Helmers and Patnam, 2011; Cunha and Heckman, 2007, 2008; Carneiro, Heckman and Vytlacil, 2011).

II. Empirics on Returns to Education

4. Diminishing marginal internal rates of return (IRR) to education (Card, 1999; Heckman, Humphries and Veramendi, 2018) and subsequent difference between average and marginal IRR to schooling (Heckman, Lochner and Todd, 2006, 2008; Heckman, Schmierer and Urzua, 2010).
5. Heterogeneity in IRR across time and skill profile (Psacharopoulos and Patrinos, 2004, 2018; Heckman, Lochner and Todd, 2008).
6. Self-selection into schooling based on realized returns (Heckman, Schmierer and Urzua, 2010; Carneiro, Heckman and Vytlacil, 2011).
7. Selection bias and sorting gains in schooling (Heckman, Schmierer and Urzua, 2010; Heckman, Humphries and Veramendi, 2018).

III. Empirics on the Job Market

8. Job market polarization and disappearing "middling jobs" (cf. e.g. Autor, Katz and Kearney, 2006; Goos and Manning, 2007; Goos, Manning and Salomons, 2009; Acemoglu and Autor, 2011; Autor and Dorn, 2013; Jaimovich and Siu, 2014; Foote and Ryan, 2015; Autor, 2015, 2019; Harrigan, Reshef and Toubal, 2016; Feng and Graetz, 2019).
9. Routine-biased technological change (cf. e.g. Autor, Levy and Murnane, 2003; Goos, Manning and Salomons, 2014; Adermon and Gustavsson, 2015; Cortes, Jaimovich and Siu, 2017)
10. Jobless recoveries following automation and simultaneous increases in productivity (Allen, 2009; Jaimovich and Siu, 2012; Ford, 2015; Graetz and Michaels, 2017; Autor and Salomons, 2018).
11. Recessions accelerating RBTC (Jaimovich and Siu, 2012; Hershbein and Kahn, 2018).
12. Dynastic human capital as the result of intergenerational transmission (Long and Ferrie, 2007, 2013; Lindahl et al., 2014, 2015; Turner et al., 2018).

IV. Growth and Productivity Puzzles

13. Supply-side secular stagnation in developed economies, i.e. the absence of expected high growth in western economies corresponding to the degree of automation and digitalization (Gordon, 2015; Acemoglu and Restrepo, 2017*b*, 2018*e*; Eggertsson, Mehrotra and Robbins, 2019).
14. Productivity puzzle: Falling labor share of national income (Elsby, Hobijn and Şahin, 2013; Piketty, 2014; OECD, 2015; Schwellnus et al., 2018) while labor productivity, total factor productivity (TFP) and value-added of labor in production is increasing (David, 1990; Madsen, 2014; OECD, 2015; Graetz and Michaels, 2018; Autor and

Salomons, 2018) and its relation to lower quality-adjusted prices of investment goods such as IT and computers (Karabarbounis and Neiman, 2014; Dao et al., 2017).

V. Empirics on Technology Adoption vis-à-vis Human Capital Endowment

15. The adoption rates of new technologies that increase productivity are low when the ratio of physical capital to human-capital-augmented labor supply is low (see e.g. Suri (2011) and Dercon and Christiaensen (2011) for results regarding agricultural technology and Caselli and Coleman (2001) and Comin and Hobijn (2004) for manufacturing and ICT).
16. More educated agents are more prone to adopting new complex technology (Comin and Hobijn, 2004; Foster and Rosenzweig, 2010).

VI. Empirics on Skill Profiles and Premiums

17. Seemingly sustained excess demand for agents educated in STEM fields (Arcidiacono, 2004; Kinsler and Pavan, 2015; Arcidiacono, Aucejo and Hotz, 2016; Deming and Noray, 2018).
18. Occupational specificity of human capital (Poletaev and Robinson, 2008; Kambourov and Manovskii, 2009*b*; Geel, Mure and Backes-Gellner, 2011; Sullivan, 2010).
19. Monetary penalization of occupational mobility (Kambourov and Manovskii, 2008, 2009*a*; Poletaev and Robinson, 2008; Autor, 2019) but less so the more skill-diversification in which an agent engages (Silos and Smith, 2015; Cortes and Gallipoli, 2017).
20. Differentiated distributional effects without economic shrinkage (Jaimovich and Siu, 2012, 2014; Acemoglu and Restrepo, 2018*b*; Caselli, 1999).
21. Convex skill-premium ordered according to routineness and conditions for it being U-shaped (Acemoglu and Autor, 2011; Autor and Salomons, 2018; Autor, 2019).

In the rest of this paper, I will continuously refer to the empirical observations documented in the introduction list by expressing the corresponding number of the specific piece of evidence in brackets. Hence, [1] refers to the first piece of evidence, and so on.

Related Literature − This study is related to six strains of modern economics: the psychometric literature on skill formation, the empirics of returns to education plus theories and empirical evidence on unbalanced growth, human capital investment and the task-based framework in addition to the third generation of distributional macromodels.

The psychometric literature offers two main empirical insights that are relevant to this study. First, the quality and magnitude of early childhood investments have large consequence for skill acquisition later in life (Almond and Currie, 2011; Graff Zivin and Neidell, 2013). Second, cognitive and non-cognitive skills complement each other - higher skill in one indicates higher productivity in acquiring the other (Cunha, Heckman and Schennach, 2010; Helmers and Patnam, 2011). Cunha et al. (2006) and Cunha and Heckman (2007, 2008) provides a conceptual framework for expressing these results: through three main concepts. First, *self-productivity* of

human capital which means that skills produced at one stage augment the ones attained at later stages. Second, *dynamic complementarity* indicating that skills produced at one stage raise the productivity of human capital investment at subsequent stages. Finally, *cross-productivity* of human capital implying that stock in one skill eases acquisition of other skills, and vice-versa. In the model presented in this study, I microfound the dynamics of skill acquisition upon these three concepts.[6]

The works of Mincer (1958, 1974) are seminal to the returns-to-schooling literature. This model builds on a distinction between schooling and experience and typically assumes a psychological (hedonic) cost to education explicated in the utility function of agents. The resulting decision rules are simple and conveniently expressed as a regression prompting their almost five-decade long popularity in the empirical literature. Mincerian regression coefficients were commonly interpreted as *internal rates of return* (IRR) to education as explicated by Becker (2009). However, around the millennial shift evidence emerged calling into question the validity of Mincer models' theoretical construction and their corresponding interpretation. For instance, Katz and Autor (1999) and Heckman, Lochner and Todd (2006) reject the Mincerian functional forms. Employing a non-parametric approach, moreover, Heckman, Lochner and Todd (2008) find two deviations from key assumptions of the Mincer model – namely parallelism and linearity in log earnings which, in turn, are quantitatively important for estimating IRRs. Furthermore, Bhuller, Mogstad and Salvanes (2017) find a negative bias in IRRs measured using the Mincer model due to non-stationarity of life-cycle earnings. Finally, Psacharopoulos and Patrinos (2004, 2018) finds heterogeneity in returns to schooling across time and skill profile.

These insights has led to the use of more generalized functional forms for analyzing returns to schooling, prompting a more critical approach. Card (1999) and Heckman, Humphries and Veramendi (2018) find diminishing marginal returns to education. In agreement with the former piece of evidence, Heckman, Lochner and Todd (2006, 2008) and Heckman, Schmierer and Urzua (2010) find difference between yields of schooling on average and margin respectively. Moreover, Heckman, Schmierer and Urzua (2010) and Carneiro, Heckman and Vytlacil (2011) document self-selection into schooling based on realized returns. Consistent with findings of skill formation literature(esp. Cunha, Heckman and Schennach, 2010), Heckman, Humphries and Veramendi (2018) illustrate selection bias and sorting gains in schooling which are akin to endowment effects in human capital for technology adoption (Comin and Hobijn, 2004; Foster and Rosenzweig, 2010). Basing the model of this study on the dynamics of skill formation with focus on endowment effects, I am thusly able to replicate a sorting behavior and return structure consistent with these empirics.

[6] As pointed out by Cunha and Heckman (2007, 2008), a consequence of these insights is the obsolescence of ability-skill and nature-nurture dichotomies which play a crucial role in a big part of applied microeconomics literature. Dichotomies are namely useful only if they are *crisp* and *exhaustive* which are here undermined by cross-productivity and dynamic complementarity of skills respectively. Subsequently, since the ability-skill distinction becomes less useful, they are not distinguished in the model and expressed under the umbrella of human capital stock for varying skills. Similarly, due to the lack of poignancy in the nature-nurture dichotomy, I focus on the endowment channel of skill acquisition – as opposed to the individuals' pedagogical capacity. The latter is assumed to be a societal variable interpreted as quality in schools and universities.

Theories of unbalanced growth were developed to explain sectoral heterogeneity in the effects of technological change such as factor-productivity and labor share of production. In this regard, works of Kuznets (1957, 1973) and Chenery (1960) were seminal why these facts are referred to as *Kuznets facts*. An example of such development is the labor shift away from agriculture and towards manufacturing during early and mid twentieth century following introduction of mechanized farming instruments (Allen, 2009; Goldin and Katz, 2009). Later studies –Kongsamut, Rebelo and Xie (2001), Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008) – aimed at unifying these observations of sectoral heterogeneity and labor transition with the seemingly contradicting aggregate empirics of constancy with regards to growth rate of output per capita, capital-output ratio, real rate of return to capital and labor-capital shares in national income. Documented in the seminal work of Kaldor (1957, 1962), these stylized observations are referred to as *Kaldor facts* in modern economic parlance.

Recent empirical findings, however, undermine several of these stylized facts such as constant interest rates and wealth-to-output ratios (Piketty, 2014; Eggertsson, Robbins and Wold, 2018) plus a constancy in labor share of production (Elsby, Hobijn and Şahin, 2013; OECD, 2015; Schwellnus et al., 2018). Nevertheless, diminishing labor share in one national economy may be due to off-shoring of production and, thus, an increase elsewhere. However, Karabarbounis and Neiman (2014) and Dao et al. (2017) rule out this scenario showing the global labor share has indeed diminished likely due to lower quality-adjusted prices of investment goods such as information and communication technology (ICT). Hence, researchers have felt the need to articulate new frameworks of growth explaining this new situation, and at the same time provide convincing reasons why the Kaldor facts are failing at this particular juncture and not previously.

As a result, a marginalized and more pessimistic growth framework has gained recent attention. This narrative was first put forth in the seminal works of Marx (1889) predicting complete eventual labor immiseration. In Marx's view, the ability of capital – or *dead labor* in his vernacular – to reproduce *labor power* entails that as surplus resulting from new – *living* – labor is saved as new stocks of capital the need for additional labor supply will diminish, leading to an exceeding precarious situation for laborers. Such insights were incorporated in the seminal input-output analysis of Leontief (1986) – widely used in the international trade literature – where labor was seen as a primary resource in production with empirical challenges distinguishing between the contribution of labor and capital in production. Most likely, however, Marx's framework would view both factor-augmenting and labor-displacing effects of technological progress as reproducing labor power, and hence not distinguish between the two – as is otherwise customary in modern economic analysis. Nevertheless, as we have seen already, the potential labor-displacing effects of shifts in technology have been widely ignored in the economic literature until just recently. Such radical departure from labor-capital substitutability is also one of the reasons for Leontief aggregator's lack of popularity other than a theoretical object.

A more widely cited work on labor immiseration vis-à-vis unbalanced growth within the economic literature is Baumol (1967) which anticipates total obsolescence of sectors undergoing automatization. Several authors have recently built on these insights in order to match – with

varying degrees of alarmism – the empirical evidence on RBTC[7], job market polarization[8] and the dissipation of middling tasks[9] in the recent two decades. For instance Berg, Buffie and Zanna (2018) find that in most conceivable cases automation has strongly positive and negative effects on growth and equality respectively – even in the optimistic case of robots only being used for a subset tasks or immunity of certain sectors to such automation technology. A more pessimistic view, Susskind (2017) predicts full immiseration of labor through a process of *task encroachment* – as 'advanced capital' accumulates the subset of tasks performed by labor diminishes over time and approaches zero. Feng and Graetz (2019) find evidence of such encroachment by studying job training requirements. They illustrate that employment have been polarized by initial occupational training requirements and shifted towards more complex occupations; and that the relationship between complexity and employment growth is weakest among occupations with low training requirements.

Another attempt at producing a theoretical framework for resolving the empirical puzzles mentioned is Acemoglu and Restrepo (2018*e*), which incidentally also is the most comprehensive work on capturing the insights of the task-based framework. Acemoglu and Restrepo (2018*e*), and the extension in Acemoglu and Restrepo (2018*c*), build on the theoretical grounds laid by Zeira (1998) and the empirically backed conceptual framework described in Autor, Levy and Murnane (2003). The task-based approach offers a more radical form of creative destruction than that of the early Schumpeterian growth framework (cf. e.g. Aghion, 2002; Aghion, Akcigit and Howitt, 2014). In the latter approach the destructive part of technological change is limited to less productive intermediate firms being replaced by ones which are more so or better at patenting their innovations. In the task-based framework, on the other hand, two kinds of destruction patterns are present: labor-skill obsolescence (as a result of automation) and automated tasks becoming obsolete (following new and ground-breaking technology), which both are potentially more ruinous than a number of firms going bust. A similar approach is made in Acemoglu and Restrepo (2018*c*) although for the distinguished low- and high-skill automating capital stocks.

The literature on human capital investment has a long tradition. The theoretical models can be classified into four main categories: *learning-by-paying*, *learning-by-doing*, *learning-by-learning* and *learning-by-imitation*.[10] Learning-by-paying models focus on the direct cost of education such as tuitions and deal with credit constraints agents face when making choices on pursuing education. Examples of these models are Becker (1962) and partly Becker and Tomes (1986). Empirical research on the importance of credit constraints for schooling decisions is carried by Lochner and Monge-Naranjo (2011) and Carneiro and Heckman (2002) among others.

[7]See Autor, Levy and Murnane (2003); Goos, Manning and Salomons (2009, 2014); Acemoglu and Autor (2011); Adermon and Gustavsson (2015); Cortes, Jaimovich and Siu (2017); Hershbein and Kahn (2018)

[8]See Autor, Katz and Kearney (2006); Goos and Manning (2007); Goos, Manning and Salomons (2009); Autor (2015); Autor and Dorn (2013); Foote and Ryan (2015); Harrigan, Reshef and Toubal (2016).

[9]See Acemoglu and Autor (2011); Autor (2013, 2019); Autor and Salomons (2018).

[10]Certain combinations of these models, moreover, are typically classified under the label *on-the-job-learning*. While one would expect this classification be the same as learning-by-doing, it often is not and contains aspects of the other models. Nevertheless, one facet that is common in models of on-the-job-learning, is that agents are working parallel with their learning decisions, however they affect their stock of human capital.

Learning-by-doing models (see for instance Kenneth J. Arrow, 1962; Stokey, 1988; Robert E. Lucas Jr., 2009) focus on experience gained by performing the tasks included in a job directly and hence highlights the importance of tenure in wage gains (examples of empirical explorations include Stevens, 2003; Loury, 2006; Krolikowski, 2017).

In learning-by-learning models, agents devote a portion of their time (Becker, 1985, 2009), attention (Becker and Murphy, 1992), human capital stock (Ben-Porath, 1967; Rosen, 1976, 1983) or a combination of the three to learning (see e.g. Heckman, 1976, for human-capital-augmented attention allocation). Subsequently, models of this type emphasize the resulting *foregone earnings* during education acquisition (some empirical studies are Heckman and Robb Jr, 1985; Card, 1999; Ginther, 2000; Chabé-Ferret, 2015).

Finally, in learning-by-imitation models agents observe and then emulate laborers with higher productivity. These approaches usually involve some temporal (attentional) (Lucas Jr and Moll, 2014) or fiscal (Boyan Jovanovic and Rafael Rob, 1989; Dasgupta, 2012; Perla and Tonetti, 2014) costs in search for finding such agents.[11] Consequently, explorations based on these models deal with the diffusion and accumulation of knowledge and their spillover effects.[12]

The model that I develop in this study has a learning-by-learning structure. Agents will divide their attention budget between labor – that will generate instantaneous income – and learning – that will be added to the stock of human capital, increase the augmented wage rate and thus indirectly provide labor gains by generating future earnings. Moreover, agents are aware that risk of skill obsolescence entails future income streams being tentative. The learning and stock schemes, as mentioned earlier, are tuned to the concepts of skill formation.

The first generation of distributional macro models were developed by considering representative agent environments. The research included new-Keynesian and real business cycle approaches to the economy. Second generation models on the other hand, incorporated some considerations with respect to *ex-post* heterogeneity caused by idiosyncratic wealth risk. Seminal papers include Bewley (1977), Aiyagari (1994), Den Haan (1997) and Krusell and Smith (1998). These studies did not improve significantly on the estimations and intuitions of the first-generation models – mainly due to their counter-empirical assumption that wealthy individuals in the economy are merely scaled versions of the less fortunate. Indeed, wealthy individuals exhibit lower marginal propensity to consume out of transitory changes in earnings while having higher saving rates out of their expected permanent income. By employing more granular data, higher computational power and advancements in the mathematical theory of mean-field games[13] several seminal papers showing the importance of heterogeneity for the macroeconomy have been published. Some of these include Lucas Jr and Moll (2014), Gabaix et al. (2016), Ahn et al. (2018) and Kaplan,

[11] Hence, these models can alternatively be thought of as *learning-by-searching*.

[12] The literature on knowledge accumulation is mainly focused on firm dynamics, but the examples given here are the ones that focus on the labor side.

[13] See Nuño and Moll (2018) and references therein.

Moll and Violante (2018). The model developed in this study will use the theoretical techniques to derive analytic and exact solutions.

Conceptually, the most related papers are Autor, Levy and Murnane (2003) and Acemoglu and Restrepo (2018*e*) on the task-based framework. I incorporate their ordering of tasks along their degree of routineness and assume comparative advantage of labor in non-routine tasks. Moreover, concepts on skill formation are borrowed from Heckman, Lochner and Todd (2006) and Heckman, Humphries and Veramendi (2018).

The mathematics are combinations of techniques partly in Stokey (1988, 2018) and partly in Lucas Jr and Moll (2014) and Nuño and Moll (2018). Considering a continuum of types in "knowledge capital" among workers in Stokey (1988) is very similar to the approach that I am proposing here. Stokey (1988) considers a representative-agent partial-equilibrium deterministic learning-by-doing scheme. In contrast, Stokey (2018) develops a deterministic general equilibrium structure yet without any mechanism for learning. I develop a heterogeneous-agent stochastic general-equilibrium approach with skill acquisition by borrowing mean-field techniques employed in Lucas Jr and Moll (2014) and Nuño and Moll (2018). Lucas Jr and Moll (2014) provides a dynamic general equilibrium leaning-by-imitation scheme knowledge accumulation with drift-jump stochasticity. However, their model considers only a single measure of productivity as proxy for human capital stock and thus does not deal with plurality of skills which is essential in dealing with risks of devaluation. Moreover, in their model stochasticity is a result of the meetings with higher productivity individuals and hence they are not facing and possibility of their skills becoming obsolete. Nuño and Moll (2018) provides a general setting for how to carry out aggregation in environments with *ex-post* idiosyncratic wealth risk.

The rest of the paper will have the following structure. To provide some intuition, in Section 2.1 I will first present an expository partial-equilibrium bivariate static model of skill diversification under risk for obsolescence. In Section 2.2 I then develop the full model – dynamic general equilibrium with a continuum of tasks. Section 2.3 provides an extension to the case with risk of skill obsolescence. In Section 2.4, the model is given a generational interpretation. Finally, in Section 3 I will summarize the key results, make some concluding remarks and put forth some future issues for further research.

2 Theory

2.1 An Expository Dichotomous Static Model of Skill Diversification

We begin with an expository model in a static two-skill environment. We assume that the agent wants to maximize expected utility. There are two skills $j \in \{1, 2\}$ generating income streams $y_j = w_j h_j \ell_j$ where w_j is the wage level, h_j is the stock of human capital and ℓ_j is (attention to) labor, all pertaining to skill j. The sum of the income streams equal consumption c. The

attention budget of the agent is given by

$$a_1 + \ell_1 + a_2 + \ell_2 = 1 \tag{1}$$

where a_j is attention to learning as stated previously. Human capital is accumulated according to the following learning function:

$$h_j = g_j(a_j)h_{j,0}, j = 1, 2, \qquad g_j(0) = 1, g_j' > 0, g_j'' < 0 \text{ for } a_j \in [0, 1], \tag{2}$$

where $h_{j,0}$ is initial stock of human capital in skill j. The characteristics assumed for the learning function $g_j' > 0$ and $g_j'' < 0$ is standard in the literature (cf. e.g. Willis, 1986; Kalemli-Ozcan, Ryder and Weil, 2000; Cunha et al., 2006).

Definition 2.1. *We adopt the following convention.*

(a) A skill j is self-productive if skills produced at one stage augment the ones attained at later stages, or formally,

$$\frac{\partial}{\partial a_j}\Delta h_j > 0$$

(b) A skill j satisfies dynamic complementarity if produced at one stage raise the productivity of human capital investment at subsequent stages, or formally,

$$\frac{\partial^2}{\partial a_j \partial h_{j,0}}\Delta h_j > 0$$

It follows promptly that the learning mechanism described in (2) is self-productive [1] and satisfies dynamic complementarity [2]. We can then readily show that returns to learning a skill is increasing but at a diminishing rate.

Proposition 2.1. *Returns to learning a skill is increasing but at a diminishing rate, that is,*

$$\frac{\partial y_j}{\partial a_j} > 0, \qquad \frac{\partial^2 y_j}{\partial a_j^2} < 0.$$

Proof. The proof is merely inserting h_j from (2) into the earnings function $y_j = w_j h_j \ell_j$ and taking the corresponding derivatives with respect to a_j. □

Hence we have shown that by assuming a learning scheme that is self-productive and dynamically complementary we obtain a learning function that satisfies empirically verified properties, namely diminishing marginal returns and subsequent discrepancy between average and marginal returns to schooling [4].

We assume now further that skill 1 is easier to learn, that is, $g_1(a) > g_2(a), a \in [0, 1]$. The independent probability of skill obsolescence is given by $\Pr(w_j = 0) = \xi_j$.

For ease of analysis we are going to use the following specification of the attention budget instead:

$$(a_1, a_2, p) \in [0, p] \times [0, 1-p] \times [0, 1] \tag{3}$$

with $p = p_1 \triangleq a_1 + \ell_1$, or equivalently, $1 - p = p_2 \triangleq a_2 + \ell_2$. In other words, p is the portion of the attention given to labor or learning of skill 1. Moreover, $\boldsymbol{p} \triangleq (p_1, p_2)$ is the agent's *job description*, which is described exhaustively through p. Hence, we will use the term job description interchangeably for p and $\boldsymbol{p}$.

Hence the maximization problem becomes

$$\max_{a_1, a_2, p} \mathbb{E}(U(c)) = U(y_1 + y_2)(1 - \xi_1)(1 - \xi_2) + U(y_1)(1 - \xi_1)\xi_2 + U(y_2)(1 - \xi_2)\xi_1 + U(0)\xi_1\xi_2, \tag{4}$$

subject to (1), or equivalently (3). The utility function U increasing in consumption at a diminishing rate ($U' > 0$ and $U'' < 0$).

We can deduce the following lemma which states that not all of attention to a particular skill will go exclusively towards learning that skill. In other words, if attention is devoted to a particular skill ($p_j \neq 0$), then some attention is given to labor with that skill ($\ell_j \neq 0$).

Lemma 2.1. *Let a_j^* be the attention to learning in skill j that maximizes expected utility in (3). Then it follows that:*

$$0 \leq a_j^* < p_j. \tag{5}$$

Using this lemma we then can prove the following proposition.

Proposition 2.2. *If job description p is fixed, the allocation of time between learning a_j and labor $\ell_j = p_j - a_j$ is independent of wage rates w_j, human capital endowments $h_{j,0}$ and obsolescence probabilities ξ_j of each skill, and only depends on the characteristics of the learning functions g_j, $j = 1, 2$.*

For proofs see Appendix A. From now on we adopt the following convention. We say that an agent *specializes in skill j* if the optimal choice of attention to learning for the other skill is zero, i.e. $a_i^* = 0$ for $i \neq j$. We say that an agent *focuses exhaustively on skill j* if the portion devoted to learning and labor with it adds up to the whole attention budget, i.e. $p_j^* = 1$.

In case the job description p is not fixed, in addition to (66) the following optimality condition holds for interior solutions:[14]

$$\frac{(1 - \xi_1)w_1 h_1}{(1 - \xi_2)w_2 h_2} = \frac{U'(y_1 + y_2)(1 - \xi_1) + U'(y_2)\xi_1}{U'(y_1 + y_2)(1 - \xi_2) + U'(y_1)\xi_2} \tag{6}$$

[14] This is the result of the following first order condition: $\frac{\partial}{\partial p}\mathbb{E}(U(c)) = 0$.

The equation above states that the optimal job description p^* is chosen such that the relative marginal gains from income in the skills equal their relative expected marginal utility. If the left-hand side of (6) is larger than the right-hand side, then the agent's optimal decision is to focus exhaustively on skill 1 and vice versa.

We can derive the following interesting comparative statics for endowment profiles $h_{j,0}$ of the agents. The proposition states that optimal attention to learning and labor with skill j is increasing in the endowment in said skill $h_{j,0}$ provided that the wage rate of the other skill is low enough.

Proposition 2.3. *Wherever differentiable,* ceteris paribus *the following holds,*

$$\frac{\partial}{\partial h_{1,0}} a_1^* \geq 0, \qquad \frac{\partial}{\partial h_{1,0}} a_2^* \leq 0, \qquad \frac{\partial}{\partial h_{1,0}} p^* \geq 0 \tag{7}$$

if and only if

$$w_2 \leq \left(1 + \frac{\xi_2}{1-\xi_2} \cdot \frac{U''(y_1)}{U''(y_1+y_2)}\right) \frac{h_1}{h_2} w_1. \tag{8}$$

Remark 2.3.1. *Similar corresponding results can be derived for $h_{2,0}$ where inequalities in (7) are reversed and (8) is replaced by*

$$w_1 \leq \left(1 + \frac{\xi_1}{1-\xi_1} \cdot \frac{U''(y_2)}{U''(y_1+y_2)}\right) \frac{h_2}{h_1} w_2. \tag{9}$$

Observe that if $\ell_j^* = 0$ then $a_j^* = 0$. From Proposition (2.3) we can then find necessary and sufficient conditions for an agent to exhaustively focus on skill 2. Of course corresponding corollary can be derived for skill 1.

Corollary 2.3.1. *The agent focuses exhaustively on skill 2 if and only if*

$$h_{2,0} > \left(1 + \frac{\xi_2}{1-\xi_2}\right) \frac{w_1}{w_2} h_1 \mid_{\ell_2^* = a_2^* = 0} . \tag{10}$$

If skill 2 safe relative to skill 1 - i.e. $\xi_2 = 0$ - then the condition becomes,

$$w_2 h_{2,0} > w_1 h_{1,0}. \tag{11}$$

Proof. The follows directly from Proposition (2.3), where (10) provides the complement set characterized by (6) evaluated at $\ell_2^* = a_2^* = 0$. Finally, (11) follows from (10) when $\xi_2 = 0$. □

Having provided characteristics for exhaustive focus on skills, we move on to derive conditions for specialization upon skills, i.e. $a_j^* = 0$. We define the following *ceteris paribus lock-in thresholds*:

$$\underline{h}_{1,0} \equiv \sup_{a_1^*=0} h_{1,0}, \qquad \bar{h}_{1,0} \equiv \inf_{a_2^*=0} h_{1,0}$$

Since the optimal labor choice for skill j is given by

$$\ell_j^* = \frac{g_j(a_j^*)}{g_j'(a_j^*)}.$$

we can deduce that these thresholds have the following structure. Namely,

$$\underline{h}_{1,0} \text{ is given by } \frac{g_1(0)}{g_1'(0)} = p^*(\underline{h}_{1,0}) \qquad \text{and} \qquad \bar{h}_{1,0} \text{ is given by } \frac{g_2(0)}{g_2'(0)} = 1 - p^*(\bar{h}_{1,0}). \tag{12}$$

It is furthermore obvious that by construction $\underline{h}_{1,0} \leq \bar{h}_{1,0}$. For the remaining analysis we assume $\xi_2 \leq \xi_1$.

Figures 1 and 2 go here.

In Figure 1 we can see these *ceteris paribus* endowment thresholds of lock-in for skill 1. To begin with, the figure illustrates the results of Proposition 2.3. We see that attention to learning and labor with skill 1 is increasing with respect to human capital endowment in said skill. Moreover, the spectrum divides agents into three types: specializers in skill 2, diversifiers and specializers in skill 1 corresponding to regions (I) , (II) and (III) respectively. An interesting dynamic, however, arises when skills are assumed to be safe, i.e. when the probabilities of obsolescence ξ_j approach zero. In that case, we can show that there are no diversifiers. This is depicted in Figure 2, where region (II) is nearly empty. We summarize the results in Proposition 2.4.

Proposition 2.4. *Let $\xi_2 \leq \xi_1$. Then $\bar{h}_{1,0} \to \underline{h}_{1,0}$ as $\xi_1 \to 0$. In other words, in absence of skill obsolescence agents purely specialize.*

Hence, direct consequence of Propositions (2.3) and (2.4) is the empirically verified results that agents sort into education based on realized returns [6], that there exists selection bias and sorting gains in schooling [7] and that human capital is specific to occupation [18]. We also see that agents which have comparatively higher endowment in skills that are more difficult to acquire, tend to be more prone to work with new complex technology [16].

Recall that returns to human capital investment is increasing at a diminishing rate which is in line with the overview of the empirical literature on returns to schooling (Card, 2001; Heckman, Humphries and Veramendi, 2018). Here we have shown that specialization is a consequence of skills being safe from obsolescence. Becker (1985) argues, however, that specialization is primarily prompted by increasing returns to education, while not considering skill obsolescence. Hence, the discussion here shows that such a conclusion is faulty, if not wholly erroneous. Indeed, even in the model presented by Becker (1985) the main driver of specialization is the skills being safe. We summarize the result in the following corollary.

Corollary 2.4.1. *Increasing returns to human capital investment is not necessary for pure specialization.*

Another interesting dynamic arises if we consider intergenerational transmission of human capital. Imagine a situation where each generation faces the same problem as in (4), while the next generation inherits the new profile stock of human capital

$$(h_{1,n,i}, h_{2,n,i}) = (g_1(a^*_{1,n,i})h_{1,n-1,i}, g_1(a^*_{2,n-1,i})h_{2,n-1,i})$$

where $n \geq 1$ is the number of generation and $i \in \{1, \ldots, N\}$ is dynasty index and N is the population size. Figure 3 shows an example where there are diversifies in the population and the skill profiles are edged at the lock-in thresholds. The example can be seen as the new introduction of a new skill - h_2 - being recently monetized, why there is less endowment variation in the population than skill 1. After five iterations however, agents are starting to diverge and after ten generations the difference is steeper. Nevertheless, there always is the intermediate population of diversifiers acting as a bridge between the groups of specializers at the edges. On the other hand, in Figure 4 the agents do not perceive any risk of obsolescence for the skills, and so very quickly the population is cloven into two distinct types. These results suggest that - given the perceived threat of human capital obsolescence - differences in relative endowment among a population have profound evolutionary implications for the division of labor in society creating distinct dynastic human capital profiles [12].

Figures 3 and 4 go here.

2.2 Full Model

We move on to a dynamic model of skill diversification with a continuum of skills where agents $i \in [0, 1]$ face the following problem

$$\max_{a_{ijt}, \ell_{ijt}, c_{it}} \mathbb{E}_0 \int_0^\infty U(c_{it}) e^{-\rho t} dt \tag{13}$$

where $j \in [N_t - 1, N_t]$ is the index of skills, N_t is the technology index frontier and the rest as discussed earlier. Income from skill j is given by,

$$y_{ijt} = w_{jt} h_{ijt} \ell_{ijt}, \tag{14}$$

where w_{jt} is the wage rate for task j at time t and h_{ijt} is the agent i's stock of human capital in skill j at time t. An agent's assets x_{it} is developed in accordance with the following law of

motion:

$$dx_{it} = (r_t x_{it} + \int_{I_t^*}^{N_t} y_{ijt} dj - p_t^c c_{it})dt + \sigma_i(x_{it}, \boldsymbol{h}_{it})dB_{it} \tag{15}$$

where r_t is the rental rate of capital, I_t^* is the task index threshold above which labor is the exclusive factor of production, p_t^c is the price of consumption, $\sigma_i(x_{it}, \boldsymbol{h}_{it})$ is an idiosyncratic volatility function, $\boldsymbol{h}_{it} = (h_{ijt})_{j \in [I_t^*, N_t]}$ is the *stock-density* functional of human capital,[15] and dB_{it} is an idiosyncratic Brownian motion. We dub $\boldsymbol{h}_{it}$ stock-density rather than density as we assume that the individual's aggregate stock of human capital h_{it} does not have to be 1. In other words,

$$h_{it} = \int_{I_t^*}^{N_t} h_{ijt} dj \geq 0. \tag{16}$$

We can then express the stock-density functional as the product of individual's aggregated stock h_i and some proper density function f_{h_i},

$$\boldsymbol{h}_{it} = h_{it} \left(f_{h_{it}}(j)\right)_{j \in [I_t^*, N_t]}, \tag{17}$$

where $f_{h_{it}}(j) = h_{ijt}/h_{it}$, so that $\int_{I_t^*}^{N_t} f_{h_{it}}(j)dj = 1$. Initial stock $\boldsymbol{h}_{i0}$ is observed by the agent. We assume the following distributional structure for the density, $f_{h_{i0}}(j) \sim e^{-\gamma_{h_i} j}$, indicating that all agents have more stock low-indexed skills than high-indexed, but relatively to-one-another at varying levels. However, we assume that the endowment aggregate stock of human capital is the same for agents in the population, i.e. $h_{i,0} = h_0 > 0$.

The law of motion for agent's human capital is given by:

$$\dot{h}_{ijt} = g_j(a_{ijt}) h_{ijt}, \tag{18}$$

where the learning function g_j satisfies the following conditions,

$$\text{(a) } g_j(0) = 0, \qquad \text{(b) } \frac{\partial}{\partial j} g_j < 0, \qquad \text{(c) } \frac{\partial}{\partial a} g_j > 0, \qquad \text{(d) } \frac{\partial^2}{\partial a^2} g_j < 0. \tag{19}$$

The requirement (19a) indicates that no learning leaves the stock of human capital unchanged and (19b) states that higher-indexed skills are more difficult to learn. The conditions (19c) and (19d) are recurrent and indicate increasing learning for any given level of attention a_{ijt} at a diminishing rate. We modify Definition (2.1) in the following manner for continuous-time framework.

Definition 2.2. *We adopt the following convention.*

(a) A skill j is self-productive if skills produced at one stage augment the ones attained at later

[15] Indeed the agent might have stock of human capital for $j \in [0, I_t^*]$ but that is irrelevant to the decision of the agent and hence to the production.

stages, or formally,

$$\frac{\partial}{\partial a_j}\dot{h}_j > 0$$

(b) *A skill j satisfies dynamic complementarity if produced at one stage raise the productivity of human capital investment at subsequent stages, or formally,*

$$\frac{\partial^2}{\partial a_j \partial h_j}\dot{h}_j > 0$$

Once again, it follows promptly that the learning mechanism described in (19) is self-productive [1] and satisfies dynamic complementarity [2]. We can then readily show that returns to learning a skill is increasing but at a diminishing rate.

Proposition 2.5. *Returns to learning a skill is increasing but at a diminishing rate, that is,*

$$\frac{\partial y_j}{\partial a_j} > 0, \qquad \frac{\partial^2 y_j}{\partial a_j^2} < 0.$$

Proof. The proof is replacing h_{ijt} with $h_{ijt^-}e^{\dot{h}_{jt}}$ into the earnings function (14), then employing (19) and finally taking the corresponding derivatives with respect to a_j. □

Hence we have again shown that by assuming a learning scheme that is self-productive and dynamically complementary we obtain a learning function that satisfies empirically verified properties, namely diminishing marginal returns and subsequent discrepancy between average and marginal returns to schooling [4]. Also due to the *ex-ante* and *ex-post* heterogeneities in relative skill stock and wealth shocks respectively, we will have corresponding heterogeneity in returns to education across skill profile and time [5].

We move on to developing the environment further. Labor's exclusive threshold of factor production I_t^* is given by,

$$I_t^* = \min\{I_t, \widetilde{I}_t\} \tag{20}$$

where I_t is the threshold below which there is available technology to perform the tasks by machines, while $\widetilde{I}_t$ is the threshold below which it is cheaper to perform the tasks by machines - provided that the technology is available. We will later on in the text specify conditions for which it holds that

$$I_t^* = I_t, \tag{21}$$

but for now we stipulate it as an assumption. The arrival of new technology I_t and index frontier N_t – new tasks which replaces old tasks – are exogenous and given by the following jump

processes:

$$dI_t = dJ_I(t), \text{ where } dJ_I(t) \sim Poi(\lambda) \cdot f_{\Delta_I}(s), f_{\Delta_I} : [0, n_t] \to \mathbb{R}_+ \tag{22}$$

$$dN_t = dJ_N(t), \text{ where } dJ_N(t) \sim Poi(\lambda) \cdot f_{\Delta_N}(s), f_{\Delta_N} : [0, 1 - n_t] \to \mathbb{R}_+ \tag{23}$$

where $n_t = N_t - I_t$. Both processes are Poisson arrivals with the same intensity $\lambda > 0$ but different jump-size densities f_{Δ_I} and f_{Δ_N}. We also correspondingly define $n_t^* = N_t - I_t^*$ and $\widetilde{n}_t = N_t - \widetilde{I}_t$.

The rates of arrival, can be endogenized by assuming a pool of scientists being allocated between research and development on automation and new-task creation as illustrated in Acemoglu and Restrepo (2018*e*). However, as explained farther in this study, given the slower rate of accumulation in human vis-à-vis physical capital, it is the institutional rate of adoption that is decisive. Moreover, Acemoglu and Restrepo (2018*e*) show that for balanced growth paths, equal arrival rate of both technologies is a necessary condition. Hence, by assuming an exogenous rate of arrival, we abstract away an immaterial mechanism while rigging the growth path in favor of being balanced. Thus, if unbalanced growth is to follow yet, in less favorable scenarios it will do so still.

Finally, the attention budget is given by,

$$\int_{I_t^*}^{N_t} (a_{ijt} + \ell_{ijt}) dj = 1. \tag{24}$$

The optimization problem (13) subject to (15), (18), (22), (23) and (24) is not in general tractably solvable. Indeed we need to choose the Bernoulli utility U, the learning functionals g_j in a way so as guarantee the ability to solve the ensuing Hamilton-Jacobi-Bellman (HJB) equation and the arising functional partial differential equations. We therefore assume the following structure:

Assumption 1.

$$U(c) = c - \frac{1}{2}\alpha c^2 \textit{ where } 0 < \alpha \lll 1, \quad \textit{and} \quad g_j(a) = e^{-\gamma_g j}(a - \frac{1}{2}\xi a^2) \textit{ where } \gamma_g > 0 \textit{ and } 0 \lll \xi < 1.$$

Assuming a linear-quadratic Bernoulli utility (felicity) function removes some quantitative nuance from the agent's saving behavior but allows us to get tractability elsewhere, namely the decisions on human capital investment. Nevertheless, this functional form imposes some limitations. It indeed implies increasing absolute risk aversion and has a point of satiation. Nevertheless, this utility could be seen as a second order Taylor approximation of the following logarithmic utility function,

$$\widetilde{U}(c) = \frac{2}{\alpha} \ln(1 + \frac{1}{2}\alpha c) = c - \frac{1}{2}\alpha c^2 + o(c^3).$$

Since the Taylor expansion of the logarithmic function has unity as radius of convergence, then

$U(c)$ in Assumption 1 is a good approximation of $\widetilde{U}(c)$ as long as

$$0 < c < 2/\alpha.$$

Hence, by choosing α small enough we guarantee that this approximation is well-motivated and at the same time hinder the agent being able to reach the point of consumption satiation. Indeed it is due to this line of argument that the constrain on α is expressed as $0 < \alpha \lll 1$, indicating a very small positive number. The linear quadratic utility function allows us to employ a linear-quadratic *ansatz* for the optimal current-value function for the corresponding HJB equation. Moreover, since it is an approximation of logarithmic utility, it renders income and substitution effects to cancel out. However, it limits us to the case of near-unit constant relative risk-aversion. Nevertheless, for a more general utility structure numerical methods such as those developed by Ahn et al. (2018), Kaplan, Moll and Violante (2018) and Nuño and Moll (2018).

The functional form of g_j satisfies self-productivity. It is adjusted so that higher-indexed skills are more difficult to learn. More attention to learning a_j, however, leads to more stock of human capital h_j for a particular skill j at a diminishing rate. As mentioned earlier in the static model, a consequence of such functional form is the empirically plausible observation of increasing returns to schooling at a diminishing rate (Card, 2001; Heckman, Humphries and Veramendi, 2018). We interpret γ_g as the state of institutional education in the economy. Higher γ_g, indicates easier overall learning due to better facilities and pedagogical capabilities of teachers, instructors and communicators of knowledge in general, and vice versa. Observe however, that as $\partial^2 g_j/\partial\gamma_g^2 > 0$, more advanced institutional quality in the education system has larger effect on higher index skills – that is, those which are more difficult to obtain.

Furthermore, we define several concepts: $w_{jt}h_{ijt}$ will be referred to as *human-capital-augmented wage rate* of skill j for agent i, and similarly $h_{ijt}\ell_{ijt}$ is the *human-capital augmented labor supply* of agent i for skill j, both at time t. Moreover, the *average human-capital-augmented wage rate* for agent i at time t is defined as:

$$\overline{wh}_{it} \equiv \frac{1}{n^*(t)} \int_{I_t^*}^{N_t} w_{j,t} h_{ijt} dj. \tag{25}$$

Finally, we assume the following heterogeneity structure for endowment in human capital: $\gamma_{h_i} \sim \text{Exp}(\varrho)$, i.e. an exponential distribution with expectation $1/\varrho$.

In order ensure the tractability of analysis, we look at the solution for an exhaustive number of agent types.

Case 1 - Manageable human-capital-augmented wage rate $w_{jt}h_{ijt} = w_{j't}h_{ij't}, \forall j, j' \in [I_t^*, N_t]$: We first derive the optimal decision functions for this agent type under the assumption of no asset and jump risks, and then move onto discuss a solution with a specific risk function. Hence, the following assumption:

Assumption 2.

$$\textit{No asset or jump risk: } \sigma_i(x_{it}, \boldsymbol{h}_{it}) = \lambda = 0.$$

The following result is then derived.

Proposition 2.6. *Under assumptions 1 and 2, the agent with manageable human-capital-augmented wage rate facing the problem in (13) subject to (15), (18), (22), (23) and (24), is a full diversifier and acts in accordance with the following decision rules:*

$$a^*_{ijt} = \frac{1}{\xi}\left(1 - \sqrt{(4e^{\gamma_g j}\xi r_t - 1)\frac{w_{jt}h_{ijt}}{\overline{wh}_{it}}}\right), \qquad \ell^*_{ijt} = \frac{1}{n^*_t} - a^*_{ijt} \tag{26}$$

$$c_{it} = \frac{1}{\alpha}\left(1 - D_{1it} - D_{2t}\widetilde{Q}\right), \text{ where } \widetilde{Q} = x_{it} + \int_{I^*_t}^{N_t} D_{ijt}\sqrt{2h_{ijt}}dj \tag{27}$$

and D_{1it}, D_{2t} and D_{ijt} are constants given by

$$D_{ijt} = \sqrt{\frac{2e^{\gamma_g j}\overline{wh}_{it}w_{ijt}}{4\xi r_t - e^{-\gamma_g j}}}, \qquad D_{2t} = \frac{\alpha(\rho - 2r_t)}{6\left(p^c_t\right)^2} \tag{28}$$

$$D_{1t} = \frac{(\rho - 2r_t)\left[1 + \frac{\alpha\overline{wh}_{it}\left(1 - \frac{1}{\xi}n^*_t\right)}{p^c_t}\right]}{p^c_t\left(4\rho - 5r_t\right)} \tag{29}$$

We also derive the decision rules for the case with following structure on asset volatility,

Assumption 2′.

$$\textit{No jump risk: } \lambda = 0, \textit{ but asset risk given by } \sigma_i(x_{it}, \boldsymbol{h}_{it}) = \sigma\sqrt{2x\left(\widetilde{Q} + \frac{D_{1t}}{2D_{2t}}\right)},$$

where $\widetilde{Q}$ is the same as in (27), but D_{ijt}, D_{1t} and D_{2t} are instead given by,

$$D_{ijt} = \sqrt{\frac{2e^{\gamma_g j}\overline{wh}_{it}w_{ijt}}{4\xi(r_t + \sigma^2) - e^{-\gamma_g j}}}, \qquad D_{2t} = \frac{\alpha(\rho - 2(r_t + \sigma^2))}{6\left(p^c_t\right)^2} \tag{30}$$

$$D_{1t} = \frac{(\rho - 2(r_t + \sigma^2))\left[1 + \frac{\alpha\overline{wh}_{it}\left(1 - \frac{1}{\xi}n^*_t\right)}{p^c_t}\right]}{p^c_t\left(4\rho - 5(r_t + \sigma^2)\right)}. \tag{31}$$

The volatility structure above is a qualified geometric diffusion. Just as the regular one, however, the volatility approaches zero as financial capital stock x diminishes indefinitely, which in turn guaranties non-negative values on assets. The particular qualifications here are in place mainly so as to achieve tractability, but nevertheless do not entail any unusual consequences. For instance, one implication of the qualifications are that agents with larger stocks of human will have higher volatility in their portfolios, which is correlationally sound. The corresponding result to Proposition 2.6 is then the following,

Proposition 2.7. *Under assumptions 1 and 2′, the agent with manageable human-capital-augmented wage rate facing the problem in (13) subject to (15), (18), (22), (23) and (24), is a full diversifier and acts in accordance with the following decision rules:*

$$a^*_{ijt} = \frac{1}{\xi}\left(1 - \sqrt{(4e^{\gamma_g j}\xi(r_t + \sigma^2) - 1)\frac{w_{jt}h_{ijt}}{\overline{wh}_{it}}}\right), \qquad \ell^*_{ijt} = \frac{1}{n^*_t} - a^*_{ijt} \tag{32}$$

$$c_{it} = \frac{1}{\alpha}\left(1 - D_{1it} - D_{2t}\widetilde{Q}\right), \tag{33}$$

where $\widetilde{Q}$ is the same as in (27), but D_{ijt}, D_{1t} and D_{2t} are instead given by (30) and (31).

Case 2 - Manageable and dominating human-capital-augmented wage rate on a subset of the index domain $w_{jt}h_{ijt} = w_{j't}h_{ij't}, \forall j, j' \in \mathcal{D} \subseteq [I^*_t, N_t]$ while $w_{jt}h_{ijt} > w_{kt}h_{ikt}, \forall j \in \mathcal{D}$ & $\forall k \in \mathcal{D}^c$:[16] Not all agents will invest in the whole index spectrum of human capital $[I^*_t, N_t]$. In fact, as illustrated later on, only measure zero of agents in the continuum will do so. Hence, we derive corresponding decision rules for agents who only invest in a subset of indexes.

Proposition 2.8. *Assume an agent has manageable and dominating human-capital-augmented wage rate on a subset of the index domain $\mathcal{D}$ with measure $\nu(\mathcal{D})$, where $\nu : \mathcal{F}([I^*_t, N_t]) \mapsto \mathbb{R}_+$ is a measure function such that $\nu(\emptyset) = 0$ and $\nu([I^*_t, N_t]) = n^*_t$ and $\mathcal{F}([I^*_t, N_t])$ is the smallest possible sigma-algebra defined on $[I^*_t, N_t]$. Then under assumptions 1 and 2′, the agent facing the problem in (13) subject to (15), (18), (22), (23) and (24), is a full diversifier on $\mathcal{D}$ and acts in accordance with the decision rules in (32) and (33) but only for $j \in \mathcal{D}$. Moreover, every expression including integration over $[I^*_t, N_t]$ is replaced by integration over $\mathcal{D}$ and n^*_t is replaced by $\nu(\mathcal{D})$. Finally, in all expressions, the average human-capital-augmented wage rate $\overline{wh}_{it}$ is replaced by*

$$\overline{wh}^{\mathcal{D}}_{it} \equiv \frac{1}{\nu(\mathcal{D})}\int_{\mathcal{D}} w_{j,t}h_{ijt}dj. \tag{34}$$

Proof. The proof is the same as for Propositions 2.6 and 2.7 but with the adjustments mentioned in the text of Proposition 2.8 above. □

As we will see, the particularities of the model yields very specific and formally tractable structures for the wage-rate-dominating subset $\mathcal{D}$.

Finally, we arrive at the situation with jump risk, but first we need to present the production side of the economy.

[16] Observe that $\mathcal{D}^c$ is the complement set of $\mathcal{D}$, i.e. $\mathcal{D}^c = [I^*_t, N_t] \setminus \mathcal{D}$.

2.2.1 Production of Goods and Tasks

Final-Goods Producers: A competitive, risk-neutral and representative final goods producer homothetically aggregates a continuum of intermediate tasks indexed by $j \in [N_t - 1, N_t]$,

$$Y_t = \left(\int_{N_t - 1}^{N_t} y_{jt}^{\frac{\varepsilon - 1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon - 1}} \tag{35}$$

where $\varepsilon > 0$ is the technical elasticity of substitution between tasks.[17] Since we are discussing division of labor in society, I follow Becker and Murphy (1992), Acemoglu (1998) and Autor, Levy and Murnane (2003) in assuming tasks are complementary to varying degrees. This is in line with the empirical findings of Dinopoulos et al. (2011) who find that at the aggregate high- and low-skilled workers are gross complements.[18] Moreover, the technology-skill complementarity is well-documented in the empirical literature.[19] Formally, this assumptions is expressed as follows:

Assumption 3.

Tasks are imperfect substitutes but gross complements of one another, i.e. $0 < \varepsilon < 1$.

Cost minimization yields then the following demand for intermediate task j:

$$y_{jt}^D = \left(\frac{p_{jt}}{P_t} \right)^{-\varepsilon} Y_t, \text{ where } P_t \equiv \left(\int_{N_t - 1}^{N_t} p_{jt}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}, \tag{38}$$

where y_{jt}^D is the demand for task j at time and p_{jt} is its price.

Task Producers: We assume that there is a one-to-one correspondence between tasks and skills. Following Acemoglu and Restrepo (2018*a*,*e*), I assume that agents use technology $\vartheta_L(j)$ together with their human-capital-augmented labor supply for each task $h_{jt}\ell_{jt}$

$$h_{jt}\ell_{jt} \equiv \int_0^1 h_{ijt}\ell_{ijt} di, \tag{39}$$

[17] The case when $\varepsilon = 1$ results in the geometric aggregator:

$$Y_t = \exp\left(\int_{N_t - 1}^{N_t} \ln y_{jt} dj \right), \tag{36}$$

and $\varepsilon = 0$ yields the Leontief aggregator:

$$Y_t = \min_{j \in [N_t - 1, N_t]} y_{jt}. \tag{37}$$

[18] See Iranzo, Schivardi and Tosetti (2008) and references therein for evidence to the contrary. Nevertheless, in this framework, low-indexed and high-indexed agents can be gross substitutes, while tasks remain gross complements.

[19] See for instance assumption A2 in Autor, Levy and Murnane (2003) for how routine and non-routine tasks are assumed to be imperfect substitutes. For more examples of technology-skill complementarity see Goldin and Katz (1998, 2009); Krusell et al. (2000); Brynjolfsson and Hitt (2003); Caselli and Coleman (2006).

to produce the corresponding task $j \in [N_t - 1, N_t]$ as $y^S_{jt} = \vartheta_L(j) h_{jt} \ell_{jt}$. Nevertheless, for tasks in the interval $[N_t - 1, I]$, machines can also produce the task as perfect substitutes using the technology $\vartheta_M(j)$ and capital devoted to that task k_{jt} in accordance with $y^S_{jt} = \vartheta_M(j) k_{jt}$.[20] Hence, we have the following production function:

$$y^S_{jt} = \begin{cases} \vartheta_M(j) k_{jt} + \vartheta_L(j) h_{jt} \ell_{jt} & \text{for } j \in [N_t - 1, I_t] \\ \vartheta_L(j) h_{jt} \ell_{jt} & \text{for } j \in [I_t, N_t] \end{cases} \tag{40}$$

Just as Acemoglu and Restrepo (2018*a*,*e*) assume that $\vartheta_L(j)$, $\vartheta_M(j)$ and $\vartheta_L(j)/\vartheta_M(j)$ are increasing in j, where the last assumption implies that labor has comparative advantage in the production of high-indexed tasks compared to machines. Observe that for tasks $j \in [N_t - 1, I]$, labor and machines are perfect substitutes so production is done with the cheaper factor. By setting demand for tasks y^D_{jt} in (38) equal to their supply y^S_{jt} provided by (40) we can derive expressions for their prices.

Lemma 2.2. *The price of tasks are given by*

$$p_{jt} = \begin{cases} P_t \left(\frac{Y_t}{\vartheta_M(j) k_{jt}} \right)^{\frac{1}{\varepsilon}} & \textit{for } j \in [N_t - 1, I^*_t] \\ P_t \left(\frac{Y_t}{\vartheta_L(j) h_{jt} \ell_{jt}} \right)^{\frac{1}{\varepsilon}} & \textit{for } j \in [I^*_t, N_t] \end{cases}$$

Finally, we explicate a subtle assumption under which we have operated so far,

Assumption 4.

Rental rate of capital is the same for all the tasks $j \in [N_t - 1, I^*_t] : r_{jt} = r_t$.

This assumption has allowed us to abstract from the investment decisions. This can be seen as all agents investing in the same stock market where a no-arbitrage condition has rendered all indexes fiscally equivalent. Thereby, Assumption 4 will yield the relative allocation of the capital stock. Observe that the following capital market clearing condition holds:

$$K_t = X_t \text{ where } K_t \equiv \int_{N_t - 1}^{I^*_t} k_{jt} dj \text{ and } X_t \equiv \int_0^1 x_{it} di. \tag{41}$$

Price of final goods p^c_t is given by the production economy's clearing condition,

$$dX_t = Y_t - p^c_t C_t \text{ where } C_t \equiv \int_0^1 c_{it} di. \tag{42}$$

Assuming perfect competition among risk-neutral task producers, we can then prove the following proposition.

Proposition 2.9. *Let the market be perfectly competitive with task producers having production function (40) and operating under Assumption 4. Then rental rate of capital and the wage rates*

[20] See Autor and Dorn (2013) for evidence on capital and labor being gross substitutes. See León-Ledesma, McAdam and Willman (2010) and references therein for evidence to the contrary.

are given by

$$r_t = P_t \left(\frac{Y_t \cdot \int_{N_t-1}^{I_t^*} \vartheta_M^{\varepsilon-1}(j)dj}{X_t} \right)^{\frac{1}{\varepsilon}} \text{ and } w_{jt} = P_t \left(\frac{Y_t}{h_{jt}\ell_{jt}} \right)^{\frac{1}{\epsilon}} \vartheta_L^{\frac{\varepsilon-1}{\varepsilon}}(j) \tag{43}$$

for $j \in [N_t - 1, I_t^*]$ *and* $j \in [I_t^*, N_t]$ *respectively. Moreover, capital dedicated to task* $j \in [N_t - 1, I_t^*]$ *is given by*

$$k_{jt} = \mathcal{P}_{jt} X_t \text{ where } \mathcal{P}_{jt} \equiv \frac{\vartheta_M^{\varepsilon-1}(j)}{\int_{N_t-1}^{I_t^*} \vartheta_M^{\varepsilon-1}(j)dj}, \tag{44}$$

where $\mathcal{P}_{jt}$ *is a density function describing the distribution of capital among tasks* $j \in [N_t - 1, I^*]$.

We derive the following corollaries of Proposition (2.9):

Corollary 2.9.1. *The price of tasks produced by capital* $j \in [N_t - 1, I_t^*]$ *is given by*

$$p_{jt} = \frac{P_t}{\vartheta_M(j)} \left(\frac{Y_t}{X_t} \int_{N_t-1}^{I_t^*} \vartheta_M^{\varepsilon-1}(j)dj \right)^{\frac{1}{\varepsilon}}$$

Proof. The proof follows promptly from inserting (44) into the expression given by Lemma 2.2. □

Corollary 2.9.2. *The portion of aggregate capital invested in the skill-index interval* $[j', j]$, $j' \leq j$ *and* $j', j \in [N_t, I_t^*]$ *at time* t *is dubbed* $\mathcal{P}_{[j',j],t}$ *and is given by*

$$\mathcal{P}_{[j',j],t} = \frac{\int_{j'}^{j} \vartheta_M^{\varepsilon-1}(z)dz}{\int_{N_t-1}^{I_t^*} \vartheta_M^{\varepsilon-1}(z)dz}.$$

Proof. This corollary is a direct consequence of (44). □

Tasks will be produced by machines if and only they are cheaper, that is,

$$p_{jt}^M < p_{jt}^L$$

which by Lemma 2.2 yields

$$\frac{\vartheta_L(j)}{\vartheta_M(j)} < \frac{k_{jt}}{h_{jt}\ell_{jt}}.$$

Hence, by (44) we have that

$$h_{jt}\ell_{jt} < \frac{\vartheta_M^{\varepsilon}(j)}{\vartheta_L(j)} \cdot \frac{X_t}{\int_{N_t-1}^{I_t^*} \vartheta_M^{\varepsilon-1}(j)dj}. \tag{45}$$

Since by assumption $\vartheta_L(j)/\vartheta_M(j)$ is increasing in j and tasks are complementary ($0 < \varepsilon < 1$), the right-hand side above is decreasing in the task index. Thereby, there exists a threshold $\tilde{I}_t$

given by

$$h_{\tilde{I}_t,t}\ell_{\tilde{I},t} = \frac{\vartheta_M^{\varepsilon}(\tilde{I}_t)}{\vartheta_L(\tilde{I}_t)} \cdot \frac{X_t}{\int_{N_t-1}^{\tilde{I}_t} \vartheta_M^{\varepsilon-1}(j)dj}. \tag{46}$$

such that for all tasks $j \in [N_t - 1, \tilde{I}_t]$ are produced by machines, provided that the technology exists, i.e. $\tilde{I}_t \geq I_t$. Recall that $I_t^* = \min\{I_t, \tilde{I}_t\}$ where I_t is the automation threshold. By (46), we have the following condition for new automation technology to be adopted:

$$h_{I_t,t}\ell_{I_t,t} < \frac{\vartheta_M^{\varepsilon}(I_t)}{\vartheta_L(I_t)} \cdot \frac{X_t}{\int_{N_t-1}^{I_t} \vartheta_M^{\varepsilon-1}(j)dj}. \tag{47}$$

We can see here that recessions in terms of a decrease in labor productivity ϑ_L will increase the comparative advantage of physical capital and hence ease automatization – i.e. routine-biased technological change [11]. Moreover, when automation occurs, productivity increases, but as some significant stock of human capital in the labor force becomes obsolete, jobless recoveries follows [10].

To Explore whether new tasks are adopted, we need to adopt some framework for intellectual property. As new tasks become available N_t, old tasks $N_t - 1$ are threatened with obsolescence. The owners to the old technology's copy-rights need to be compensated if the new tasks pose any infringement. Ponder a situation where the owners need to be fully compensated. Then the new technology will not shrink the economy, that is,

$$y_{N_t,t} > y_{N_t-1,t} \tag{48}$$

which by (40) and (44) yields

$$h_{N_t,t}\ell_{N_t,t} > \frac{\vartheta_M^{\varepsilon}(N_t - 1)}{\vartheta_L(N_t)} \cdot \frac{X_t}{\int_{N_t-1}^{I_t^*} \vartheta_M^{\varepsilon-1}(j)dj}. \tag{49}$$

Now consider a situation where there are no laws protecting intellectual properties. In such a case new technology is adopted if it is cheaper, risking shrinkage of the economy. In other words,

$$p_{N_t,t} < p_{N_t-1,t} \tag{50}$$

which by Lemma 2.2 and (44) yield once again the condition in (49). This is of course a result of assuming perfect competition. As the two extreme frameworks of intellectual property yield the same adoption conditions, all the intermediate cases will as well.

Comparing conditions for adoption of automation (47) and new tasks (49) we see that capital accumulation - i.e. an increase in X_t - has inverse effects. Indeed, capital accumulation increases the possibility of automation adoption by reducing the price of capital, while it has the inverse effect on the adoption of new tasks through the same mechanism [15]. Moreover, new

technology – automation or new tasks – is adopted if it offers higher productivity [13]. Furthermore, as we will see, the faster rate of physical capital accumulation will entail that automation technology is more readily adopted than new tasks [9].

Finally, we can employ (43) in order to derive the relative skill premiums for indexes $k > j$,

$$\frac{w_{kt}}{w_{jt}} = \left[\underbrace{\left(\frac{\vartheta_L(j)}{\vartheta_L(k)}\right)^{1-\varepsilon}}_{\text{Productivity premium}} \cdot \underbrace{\frac{h_{jt}\ell_{jt}}{h_{kt}\ell_{kt}}}_{\text{Scarcity premium}} \right]^{\frac{1}{\varepsilon}}. \tag{51}$$

Since labor has comparative advantage in higher-indexed skills ($\vartheta_L(j)$ is increasing in j) and tasks are imperfect complements ($0 < \varepsilon < 1$), the productivity premium is less than one. Hence, a more appropriate word is productivity *discount*, rather than premium. Thus, whether wages of higher-indexed tasks are higher than lower-indexed ones depends on the relative scarcity of their human-capital-augmented labor supply. In other words, the wage rate of higher-indexed tasks is larger if and only if their corresponding human-capital-augmented labor supply is much scarcer - or more precisely - scarcer by more than a factor of the productivity premium.

An empirical observation that can be explained by the insights following (51), is the seemingly sustained excess demand for educated agents in STEM fields [17]. STEM fields are mostly non-routine and therefore high-indexed. Given the productivity discount (less than one), only scarcity will keep wages high enough to motivate agents with proper endowment profiles to pursue accumulating capital in high-indexed skills.

Multiplying (43) with the agent i's stock of human capital in skill j – h_{ijt} – and performing a logarithmic transform provides us with another interesting result – namely a richer alternative to the famous Roy (1951) model of occupation self-selection and earnings inequality in the empirical labor literature.[21] The formulation in this study is superior to the Roy model in four main ways. Firstly – and most importantly – it provides an equation that incorporates general equilibrium effects. Secondly, it is microfounded from an agent's decision-making over consumption, learning and labor, rather than only being based on stock human capital and wage rates. Thirdly, the equation captures dynamic effects, rather than having a static formulation. Finally, it is formulated for the case with polychotomous choice – a continuum of choices in fact – while remaining analytically tractable.[22] Nevertheless, this model just as the Roy model predicts self-selection into occupations based on realized returns [6], but with a more nuanced channel of dynamics. The self-selection in the Roy model is based on comparative advantage in stock of skills while here it is a consequence of both comparative advantage in skill endowment and the productivity of their corresponding task-producing technologies.

[21] For a general overview of the relevance of the Roy model in the empirical literature see Heckman and Taber (2010). Heckman and Honoré (1990) provides a detailed theoretical exploration of the dichotomous Roy model.

[22] See Dahl (2002); Rothschild and Scheuer (2013); Yamaguchi (2012) for discrete polychotomous extensions to the Roy model.

2.3 Full Model under Risk of Obsolescence

Before discussing an expression for the scarcity premium in (51), however, we derive the decision function under jump risk. We derive the following two lemmas. Lemma 2.3 provides conditions for agents having manageable human-capital-augmented wage rate. Lemma 2.4 derives a lower bound for the agents optimal current-value loss under obsolescence. Hence, by employing this lower bound in the derivation of agents' decision rules we are assuming that the behavior is slightly more cautious than optimal and therefore formally are finding a so called *supersolution* to the optimization problem.

Lemma 2.3. *Under positive jump risk $\lambda > 0$, the agent has manageable human-capital-augmented wage rate if and only if*

$$w_{jt}h_{ijt}F_{I_t+\Delta_{I_t}}(j) = w_{j't}h_{ij't}F_{I_t+\Delta_{I_t}}(j') \text{ for all } j, j' \in [I_t^*, N_t]. \tag{52}$$

Moreover, the agent has manageable and dominating human-capital-augmented wage rate on a subset of the index domain $\mathcal{D} \subseteq [I_t^*, N_t]$ if and only if

$$\begin{aligned} &w_{jt}h_{ijt}F_{I_t+\Delta_{I_t}}(j) = w_{j't}h_{ij't}F_{I_t+\Delta_{I_t}}(j') \text{ for all } j, j' \in \mathcal{D}, \\ &\qquad \text{while } w_{jt}h_{ijt}F_{I_t+\Delta_{I_t}}(j) > w_{kt}h_{ikt}F_{I_t+\Delta_{I_t}}(k), \text{ for all } j \in \mathcal{D} \text{ and for all } k \in \mathcal{D}^c. \end{aligned} \tag{53}$$

Lemma 2.4. *Assume the optimal value-function has the same structure as the* asatz *in (96). Then, under positive jump risk $\lambda > 0$, the following holds*

$$\lambda \left[\int_0^{1-n_t^*} (V(., N+s) - V) f_{\Delta_{N_t}}(s) ds \right] \geq -\lambda \mathcal{P}_{\Delta_{N_t}} x_{it}(D_{1t} + 2D_{2t}\widetilde{Q}), \tag{54}$$

where

$$\mathcal{P}_{\Delta_{N_t}} \equiv \mathcal{P}_{[N_t-1, N_t-1+\mathbb{E}\,\Delta_{N_t}],t}. \tag{55}$$

Proof. When new tasks arrive, the portion of assets given by (55) is going to become obsolete. Hence, by (96) and (27), then we have that

$$\int_0^{1-n_t^*} (V(., N+s) - V) f_{\Delta_{N_t}}(s) ds = -\mathcal{P}_{\Delta_{N_t}} x_{it} \left(D_{1t} + 2D_{2t} \left(\mathcal{P}_{\Delta_{N_t}} x_{it} + \int_{I_t^*}^{N_t} D_{ijt}\sqrt{2h_{ijt}} dj \right) \right). \tag{56}$$

Since $0 \leq \mathcal{P}_{\Delta_{N_t}} \leq 1$, (54) follows promptly. □

We now make explicit the risk and volatility structure under non-zero probability of skill obsolescence.

Assumption 2″.

$$\textit{Jump intensity } \lambda > 0, \textit{ and asset risk given by } \sigma_i(x_{it}, \boldsymbol{h}_{it}) = \sigma\sqrt{2x\left(\widetilde{Q} + \frac{D_{1t}}{2D_{2t}}\right)},$$

where $\widetilde{Q}$ is the same as in (27), but D_{ijt}, D_{1t} and D_{2t} are instead given by,

$$D_{ijt} = \sqrt{\frac{2e^{\gamma_g j}\overline{wh}_{it} w_{ijt} F_{\Delta_{I_t}(j)}}{4\xi(r_t + \sigma^2 - \lambda\mathcal{P}_{\Delta_{N_t}}) - e^{-\gamma_g j}}}, \tag{57}$$

$$D_{2t} = \frac{\alpha(\rho - 2(r_t + \sigma^2 - \lambda\mathcal{P}_{\Delta_{N_t}}))}{6\left(p_t^c\right)^2} \tag{58}$$

$$D_{1t} = \frac{\left(\rho - 2(r_t + \sigma^2 - \lambda\mathcal{P}_{\Delta_{N_t}})\right)\left[1 + \frac{\alpha\overline{wh}_{it}\left(1 - \frac{1}{\xi}n_t^*\right)}{p_t^c}\right]}{p_t^c\left(4\rho - 5(r_t + \sigma^2 - \lambda\mathcal{P}_{\Delta_{N_t}})\right)}. \tag{59}$$

with

$$\mathcal{P}_{\Delta_{N_t}} \equiv \mathcal{P}_{[N_t - 1, N_t - 1 + \mathbb{E}\,\Delta_{N_t}], t}. \tag{60}$$

We are now able to deduce the decision rules under non-zero risk for skill obsolescence. Propositions 2.10 and 2.11 provides the optimal behavior for manageable and dominating human-capital-augmented wage rate on the whole and a subset of the skill-index spectrum respectively.

Proposition 2.10. *Under assumptions 1 and 2″, the agent with manageable human-capital-augmented wage rate facing the problem in (13) subject to (15), (18), (22), (23) and (24), is a full diversifier and acts in accordance with the following decision rules:*

$$a_{ijt}^* = \frac{1}{\xi}\left(1 - \sqrt{(4e^{\gamma_g j}\xi(r_t + \sigma^2 - \lambda\mathcal{P}_{\Delta_{N_t}}) - 1)\frac{w_{jt}h_{ijt}F_{\Delta_{I_t}}(j)}{\overline{wh}_{it}}}\right), \qquad \ell_{ijt}^* = \frac{1}{n_t^*} - a_{ijt}^* \tag{61}$$

$$c_{it}^* = \frac{1}{\alpha}\left(1 - D_{1it} - D_{2t}\widetilde{Q}\right), \tag{62}$$

where $\widetilde{Q}$ is the same as in (27), but D_{ijt}, D_{1t} and D_{2t} are instead given by (57), (58) and (59).

Proposition 2.11. *Assume an agent has manageable and dominating human-capital-augmented wage rate on a subset of the index domain $\mathcal{D}$ with measure $\nu(\mathcal{D})$, where $\nu : \mathcal{F}([I_t^*, N_t]) \mapsto \mathbb{R}_+$ is a measure function such that $\nu(\emptyset) = 0$ and $\nu([I_t^*, N_t]) = n_t^*$ and $\mathcal{F}([I_t^*, N_t])$ is the smallest possible σ-algebra defined on $[I_t^*, N_t]$. Then under assumptions 1 and 2″, the agent facing the problem in (13) subject to (15), (18), (22), (23) and (24), is a full diversifier on $\mathcal{D}$ and acts in accordance with the decision rules in (61) and (62) but only for $j \in \mathcal{D}$. Moreover, every expression including integration over $[I_t^*, N_t]$ is replaced by integration over $\mathcal{D}$ and n_t^* is replaced by $\nu(\mathcal{D})$. Finally, in all expressions, the average human-capital-augmented wage rate $\overline{wh}_{it}$ is replaced by (34).*

Proof. The proof is the same as for Propositions 2.10 but with the adjustments mentioned in the text of Proposition 2.11 above. □

We need to assume (local) cross-productivity in skill formation [3] to have meaningful observations on growth paths. We adopt the following convention.

Definition 2.3. *Let $\mathcal{D}_{it}$ be the convex set of dominating and manageable skills for agent i at time t. We say that the agent is cross-productive with distance $\varsigma > 0$ in their skill if there exists a ball with radius ς at the boundary points of $\mathcal{D}_{it}$ where the stock-density function $\boldsymbol{h}_{it}$ extends along its analytic continuation.*

Now we use the assumptions we made about the individuals' aggregate stock of human capital $\boldsymbol{h}_{it}$ in (16) and (17), their skill density $f_{h_{it}}(j) \sim e^{-\gamma_{h_i} j}$ and the population's distribution in the endowment parameter $\gamma_{h_i} \sim \text{Exp}(\varrho)$. We also assume the following on the technology structure to make the illustration of the rest of analysis easier, but the arguments hold regardless.

Assumption 5.

$$\vartheta_L(j) = e^{\gamma_L j} \text{ and } \vartheta_M(j) = e^{\gamma_M j} \text{ where } \gamma_L > \gamma_M, \text{ and moreover } F_{\Delta_I} \sim e^{\delta_I j}.$$

Then by (61) and (43) we get

$$a^*_{ijt} \sim \frac{1}{\xi}\left(1 - \sqrt{e^{(\gamma_g + \left(\frac{\varepsilon-1}{\varepsilon}\right)\gamma_L + \delta_I - \gamma_{h_i})j}(4\xi(r_t + \sigma^2 - \lambda\mathcal{P}_{\Delta_{N_t}}) - e^{-\gamma_g j})P_t\left(\frac{Y_t}{h_{jt}\ell_{jt}}\right)^{\frac{1}{\epsilon}}\frac{h_0}{\overline{wh}_{it}}}\right). \tag{63}$$

Hence, we have that there exists a threshold $\widetilde{\gamma}_{h_i}$ in the population given by

$$\widetilde{\gamma}_{h_i} = \delta_I + \left(\frac{\varepsilon - 1}{\varepsilon}\right)\gamma_L + \gamma_g \tag{64}$$

where agent's with lower endowment parameter $\gamma_{h_i} < \widetilde{\gamma}_{h_i}$ will have manageable and dominating human-capital-augmented wage rate on a subset of the index domain $\mathcal{D} = [I^*_t, \iota_i), \iota_i \in [I^*_t, N_t]$. Similarly, agents with higher endowment parameter $\gamma_{h_i} \geq \widetilde{\gamma}_{h_i}$ have manageable and dominating human-capital-augmented wage rate on the subset $\mathcal{D}^c$. Let's dub these agents low and high-indexed respectively. There will be within-type heterogeneity leading to differentiated distributional effects of technology adoption within and without groups [20]. A direct consequence is then convex scarcity and subsequent wage premiums [21].

As in the static model, agents sort into education based on realized returns [6], that there exists selection bias and sorting gains in schooling [7] and that human capital is specific to occupation [18]. Observe also that low-index agents are more prone to having to change the index interval upon which they invest in human capital. Hence, they are more occupationally mobile, and penalized for it due to their skills at the automation threshold becoming obsolete [19]. We also see that agents which have comparatively higher endowment in skills that are more difficult to acquire, tend to be more prone to work with new complex technology [16].

Since the distribution of the endowment parameter is given by $\gamma_{h_i} \sim \text{Exp}(\varrho)$, there are more of low-indexed agents than high indexed ones and thereby the scarcity premium in (51) is larger than one, making the skill premium ambiguous. Observe that Assumption 5 is not crucial to the

analysis above, but provides ease in the characterization of the threshold $\widetilde{\gamma}_{h_i}$.

The resulting division into high- and low-index types of agents speaks to job-market polarization. Moreover, as physical capital is accumulated the discrepancies between groups will be exacerbated prompting the dissipation of middling jobs [8]. Moreover, due to faster accumulation of physical capital we have faster adoption rates of automated technology than new tasks leading to routine-biased technological change [9].

2.4 Intergenerational Transmission of Human Capital

We now extend our analysis to overlapping generations (OLG) of workers, who face a hazard of dying and are replaced by new generations, as in Yaari (1965) and Blanchard (1985). With the same intensity of a Poisson arrival of η, old workers die and new workers are born so that the total population is stable, and normalized to 1. The results can be extended to the case of non-constant death hazard over the life cycle as in Calvo and Obstfeld (1988). We further assume that the wealth of deceased workers is transmitted to the surviving generations though bequests or perfect annuity markets. Then, the wealth distribution will be unchanged. The line of argument so far is the same as Itskhoki and Moll (2019). For human capital we assume that the density f_h is passed on exactly as it is while the only a fraction υ of the aggregate stock reaches the next generation. This problem is equivalent to starting with a fraction υ of the initial human capital stock h_0, and thus a normalization to one will make the problem once again equivalent to one with infinitely lived agents with qualified discount factors $\rho+\eta$. A consequence of this interpretation is the perpetuation of agent-types (low- and high-indexed) through dynastic human capital [12]. Observe the crucial role of cross-productivity in skill formation [3].

3 Concluding Remarks

This study includes considerations of human capital investment based on the conceptual framework of the psychometric literature on skill formation in a task-based environment. Moreover, focus of the study is shifted towards institutional adoption of new technologies, rather than their arrival rate. In doing so, the model produces a labor immiseration scenario consistent with around a dozen pivotal empirical findings on returns to education and consequences of automation while solving several puzzles in the literature. The main mechanism for many of these empirics is shown to be the faster accumulation rate of physical vis-à-vis human capital. Other contributions of this study are providing a richer alternative to the Roy model of occupational choice and earnings inequality in the empirical labor literature and a unification of the task-based framework with the third generation of distributional models in macroeconomics fused with decisions on human capital investment. The model presented here matches the empirical observations of several strains of literature qualitatively. Quantitative matching will, however, be difficult due to simplifying assumptions regarding the felicity function. Further research should include numerical calibration with more nuanced instantaneous utility considerations.

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Appendix A - Proofs

Proofs on the Expository Static Model

Proof of Lemma 2.1. We know that $c = y_1 + y_2$ and

$$y_1 = w_1 h_1 (p - a_1), \qquad y_2 = w_2 h_2 (1 - p - a_2).$$

Then by 2 we have that

$$\frac{\partial}{\partial a_j} y_j(a_j) = w_j h_{j,0} \left(g_j'(a_j)(p_j - a_j) - g_j(a_j) \right), \tag{65}$$

where $p_1 = p$ and $p_2 = 1 - p$. Therefore, $\frac{\partial}{\partial a_j} y_j(p_j) < 0$. Moreover,

$$\frac{\partial^2}{\partial a_j^2} y_j(a_j) = w_j h_{j,0} (g_j''(a_j)(p_j - a_j) - 2g_j'(a_j)) < 0.$$

By $\frac{\partial}{\partial a_j} y_j(p_j) < 0$ and $\frac{\partial^2}{\partial a_j^2} y_j(a_j) < 0$ we get (5). □

Proof of Proposition 2.2. Since job description is fixed expected utility is maximized when income streams from the skills $y_j, j = 1, 2$ are maximized. Hence, maximization is independent of the obsolescence probabilities. By (65), we see that $w_j h_{j,0}$ is immaterial to the optimization and acts as a scaling constant. If there is an interior solution it is given by,

$$\frac{\partial}{\partial a_j} \mathbb{E}(U(c)) = 0 \Rightarrow \frac{\partial}{\partial a_j} y_j = 0$$

which yields

$$\frac{g_j'(a_j^*)}{g_j(a_j^*)} = \frac{1}{p_j - a_j^*} \tag{66}$$

and thus the choice of attention to learning a_j^* only depends on the learning function g_j. Otherwise, by Lemma 2.1, $a_j^* = 0$. Thus, the proposition is proven. □

Proof of Proposition 2.3. In order to prove this result we resolve the problem (4) with the secondary condition 1. The Lagrangian is given by:

$$\mathcal{L} = \mathbb{E}(U(c)) - \mu \left(a_1 + \ell_1 + a_2 + \ell_2 - 1 \right)$$

where μ is the Lagrangian multiplier. First-order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial x_j} = 0 \Rightarrow (1 - \xi_j) \frac{\partial y_j}{\partial x_j} \left[U'(y_1 + y_2)(1 - \xi_j) + U'(y_j)\xi_j \right] = \mu \text{ where } x_j \in \{a_j, \ell_j\}. \tag{67}$$

where

$$\text{(a) } \frac{\partial y_j}{\partial a_j} = w_j g_j'(a_j) h_{j,0} \ell_j \qquad \text{(b) } \frac{\partial y_j}{\partial \ell_j} = w_j g_j(a_j) h_{j,0} \tag{68}$$

Using (67) for a_j and ℓ_j we get,

$$\ell_j^* = \frac{g_j(a_j^*)}{g_j'(a_j^*)}. \tag{69}$$

Differentiating (69) with respect to $h_{1,0}$ yields

$$\frac{\partial \ell_j^*}{\partial h_{1,0}} = \frac{(g_j'(a_j^*))^2 - g_j(a_j^*)g_j''(a_j^*)}{(g_j'(a_j^*))^2} \cdot \frac{\partial a_j^*}{\partial h_{1,0}} \tag{70}$$

which imply

$$\operatorname{sign}\left(\frac{\partial \ell_j^*}{\partial h_{1,0}}\right) = \operatorname{sign}\left(\frac{\partial a_j^*}{\partial h_{1,0}}\right) \tag{71}$$

since $g'' < 0$. Differentiating (1) at the optimum with respect to $h_{1,0}$ we get:

$$\frac{\partial}{\partial h_{1,0}}(a_1^* + \ell_1^*) = -\frac{\partial}{\partial h_{1,0}}(a_2^* + \ell_2^*) \tag{72}$$

where $a_j^* + \ell_j^* = p_j^*$. By (70) and (72) we get

$$\operatorname{sign}\left(\frac{\partial a_1^*}{\partial h_{1,0}}\right) = -\operatorname{sign}\left(\frac{\partial a_2^*}{\partial h_{1,0}}\right). \tag{73}$$

Finally, we differentiate (67) for a_1 and ℓ_1 with respect to $h_{1,0}$ and setting the left-hand sides equal to each other and inserting (68) we get the following

$$\frac{\partial}{\partial h_{1,0}}(a_1^* + \ell_1^*) = -\frac{w_1 g_1(a_1^*)\ell_1^* \overbrace{[U''(y_1+y_2)(w_1h_1 - w_2h_2)(1-\xi_1)(1-\xi_2) + (1-\xi_1)\xi_2 U''(y_1)]}^{\equiv Q}}{U''(y_1+y_2)(w_1h_1 - w_2h_2)^2(1-\xi_1)(1-\xi_2) + (1-\xi_2)\xi_1 U''(y_2)(w_2h_2)^2}.$$

Since $U'' < 0$, $\frac{\partial}{\partial h_{1,0}}(a_1^* + \ell_1^*) > 0$ if and only if $Q > 0$, which yields the condition (8). Then, (7) follows by (71), (72) and (73). Equalities in (7) holds when the equality in (8) holds. □

Proof of Proposition 2.4. The proof mainly follows from the first-order condition (6). First, we consider the condition at $\underline{h}_{1,0}$ which yields,

$$\frac{w_1 g_1(0)\underline{h}_{1,0}}{w_2 g_2(a_2^*)h_{2,0}} = 1.$$

Since $g_1(0) = 1$ by (2), we have that

$$\frac{\underline{h}_{1,0}}{h_{2,0}} = \frac{w_2}{w_1} \cdot g_2(a_2^*).$$

Once again by (2), we have that $g_2(0) = 1$ and $g_2' > 0$, and hence $g_2(a_2^*) \geq 1$. It follows then that

$$\frac{\underline{h}_{1,0}}{h_{2,0}} \geq \frac{w_2}{w_1}. \tag{74}$$

Similarly the condition (6) at $\bar{h}_{1,0}$ yields

$$\frac{\bar{h}_{1,0}}{h_{2,0}} = \frac{w_2}{w_1} \cdot \frac{1}{g_1(a_1^*)},$$

and hence

$$\frac{\bar{h}_{1,0}}{h_{2,0}} \leq \frac{w_2}{w_1}. \tag{75}$$

Then by (74) and (75) we have that $\bar{h}_{1,0} \leq \underline{h}_{1,0}$. However, by construction $\underline{h}_{1,0} \leq \bar{h}_{1,0}$. Hence, $\underline{h}_{1,0} = \bar{h}_{1,0}$, and the proposition is proven. □

3.1 Proofs on the Full Model

Proof of Proposition 2.6. For ease of notation we suppress the agent and time indexes i and t and assume that the agent takes r_t and w_t as given. Then the general problem (13) is autonomous with the following HJB equation:

$$\begin{aligned}\rho V = \max_{a_j,\ell_j,c} \Bigg\{ & U(c) + \frac{\partial V}{\partial x}\left(rx + \int_{I^*}^{N_t} w_j h_j \ell_j dj - p^c c\right) + \int_{I^*}^{N_t} \frac{\partial V}{\partial h_j} g_j(a_j) h_j dj + \frac{1}{2}\sigma_i^2(x_{it}, \boldsymbol{h}_{it}) \frac{\partial^2 V}{\partial x^2} \\ & + \lambda \left[\int_0^{n_t^*} (V(., I^* + s, .) - V) f_{\Delta_I}(s) ds + \int_0^{1-n_t^*} (V(., N+s) - V) f_{\Delta_N}(s) ds \right] \Bigg\}\end{aligned} \tag{76}$$

where $V \equiv V(x, \boldsymbol{h}, I^*, N)$ is the optimal current-value function. The Lagrangian for the right-hand side is then given by:

$$\begin{aligned}\mathcal{L} = U(c) + & \frac{\partial V}{\partial x}\left(rx + \int_{I^*}^{N_t} w_j h_j \ell_j dj - p^c c\right) + \int_{I^*}^{N_t} \frac{\partial V}{\partial h_j} g_j(a_j) h_j dj + \frac{1}{2}\sigma_i^2(x_{it}, \boldsymbol{h}_{it}) \frac{\partial^2 V}{\partial x^2} \\ & + \lambda \left[\int_0^{n_t^*} (V(., I^* + s, .) - V) f_{\Delta_I}(s) ds + \int_0^{1-n_t^*} (V(., N+s) - V) f_{\Delta_N}(s) ds \right] \\ & - \mu \left(\int_{I_t^*}^{N_t} (a_j + \ell_j) dj - 1 \right)\end{aligned} \tag{77}$$

where μ is the Lagrangian multiplier. To find the stationary points we differentiate $\mathcal{L}$ with respect to a_j, ℓ_j and c and set the result equal to zero. For differentiation with respect to a_j and ℓ_j we need to employ the Euler-Lagrange equation for the index dimension j since we are optimizing a functional.[23] Thus, we get the following equations,

$$U'(c) = p^c \frac{\partial V}{\partial x} \tag{78}$$

$$\frac{\partial V}{\partial h_j} g_j'(a_j) h_j = \mu \tag{79}$$

$$\frac{\partial V}{\partial x} w_j h_j = \mu \tag{80}$$

[23] One could indeed optimize using different types of functional derivatives given some perturbation and provide a more granular proof for the differentiation as done by e.g. Lucas Jr and Moll (2014). Nevertheless, the Euler-Lagrange equation is indeed itself proven using directional perturbation derivatives. Hence, the application of said equation is well-motivated even though usually it is formulated along the time dimension.

derived from $\partial \mathcal{L} / \partial c = 0$, $\partial \mathcal{L} / \partial a_j = 0$ and $\partial \mathcal{L} / \partial \ell_j = 0$ respectively. First-order conditions (79) and (80) need to holds for all $j \in [I_t^*, N_t]$. Observe that since the human-capital-augmented wage rate for this type of agent is manageable ($w_{jt} h_{ijt} = w_{j't} h_{ij't}, \forall j, j' \in [I_t^*, N_t]$), we have that:

$$w_j h_j = \overline{wh} \text{ for all } j \in [I_t^*, N_t] \tag{81}$$

and therefore (80) trivially holds. Then by (79) and (80) we have that

$$\frac{g_j'(a_j)}{g_k'(a_k)} = \frac{w_j}{w_k} \cdot \frac{\frac{\partial V}{\partial h_k}}{\frac{\partial V}{\partial h_j}} \tag{82}$$

for $j \neq k$. By Assumption 1, (78), (79) and (80) yield

$$c^* = \frac{1}{\alpha} \left(1 - p^c \frac{\partial V}{\partial x} \right) \tag{83}$$

$$a_j = \frac{1}{\xi} \left(1 - e^{\gamma_g j} w_j \frac{\frac{\partial V}{\partial x}}{\frac{\partial V}{\partial h_j}} \right) \tag{84}$$

respectively. Then by (82) and (84) we get

$$a_j = \frac{1}{\xi} \left(1 - \frac{w_j}{w_k} \cdot \frac{\frac{\partial V}{\partial h_k}}{\frac{\partial V}{\partial h_j}} (1 - \xi a_k) e^{-\gamma_g (k-j)} \right). \tag{85}$$

Hence the agent's total attention to learning is given by,

$$\int_{I^*}^{N} a_j dj = \frac{1}{\xi} \left(n^* - \frac{e^{-\gamma_g k}}{w_k} \cdot \frac{\partial V}{\partial h_k} (1 - \xi a_k) \int_{I^*}^{N} \frac{e^{\gamma_g j}}{\frac{\partial V}{\partial h_j}} dj \right) \tag{86}$$

and total attention to labor - by the budget (24) - is given by

$$\int_{I^*}^{N} \ell_j dj = 1 - \int_{I^*}^{N} a_j dj. \tag{87}$$

Inserting (85) for skill k into (86) we arrive at the following expression for total attention to learning:

$$\int_{I^*}^{N} a_j dj = \frac{1}{\xi} \left(n^* - \frac{\partial V}{\partial x} \int_{I^*}^{N} \frac{e^{\gamma_g j} w_j}{\frac{\partial V}{\partial h_j}} dj \right) \tag{88}$$

which indicates that optimal attention to learning for skill is the following:

$$a_j^* = \frac{1}{\xi} \left(1 - e^{\gamma_g j} w_j \frac{\frac{\partial V}{\partial x}}{\frac{\partial V}{\partial h_j}} \right) \tag{89}$$

and thereby

$$g_j(a_j^*) = \frac{-e^{\gamma_g j}}{2\xi}\left(1 - \left(e^{\gamma_g j} w_j \frac{\frac{\partial V}{\partial x}}{\frac{\partial V}{\partial h_j}}\right)^2\right) \tag{90}$$

Thus we have that

$$\int_{I^*}^{N_t} \frac{\partial V}{\partial h_j} g_j(a_j^*) h_j dj = \frac{1}{2\xi}\int_{I^*}^{N}\left(e^{-\gamma_g j} h_j \frac{\partial V}{\partial h_j} - \overline{wh} w_j e^{\gamma_g j} \frac{\left(\frac{\partial V}{\partial x}\right)^2}{\frac{\partial V}{\partial h_j}}\right) dj. \tag{91}$$

Furthermore, total labor income by (81), (87) and (88) is given by

$$\int_{I^*}^{N} w_j h_j \ell_j^* dj = \overline{wh}\left(1 - \frac{1}{\xi}\left(n^* - \frac{\partial V}{\partial x}\int_{I^*}^{N} \frac{e^{\gamma_g j} w_j}{\frac{\partial V}{\partial h_j}} dj\right)\right). \tag{92}$$

Moreover, by Assumption 1 and (83) we have that

$$U(c^*) = \frac{1}{2\alpha}\left(1 - \left(p^c \frac{\partial V}{\partial x}\right)^2\right) \tag{93}$$

Finally, inserting (83), (91), (92) and (93) into the HJB equation (76) yields then following partial integro-differential equation (PIDE):

$$\begin{aligned}\rho V = {} & \frac{1}{2\alpha}\left(1 - \left(p^c \frac{\partial V}{\partial x}\right)^2\right) + \frac{\partial V}{\partial x}\left[rx + \overline{wh}\left(1 - \frac{1}{\xi}\left(n^* - \frac{\partial V}{\partial x}\int_{I^*}^{N} \frac{e^{\gamma_g j} w_j}{\frac{\partial V}{\partial h_j}} dj\right)\right) - \frac{p^c}{\alpha}\left(1 - p^c \frac{\partial V}{\partial x}\right)\right] \\ & + \frac{1}{2\xi}\int_{I^*}^{N}\left(e^{-\gamma_g j} h_j \frac{\partial V}{\partial h_j} - \overline{wh} w_j e^{\gamma_g j} \frac{\left(\frac{\partial V}{\partial x}\right)^2}{\frac{\partial V}{\partial h_j}}\right) dj + \frac{1}{2}\sigma_i^2(x_{it}, \boldsymbol{h}_{it}) \frac{\partial^2 V}{\partial x^2} \\ & + \lambda\left[\int_0^{n_t^*} (V(., I^* + s, .) - V) f_{\Delta_I}(s) ds + \int_0^{1-n_t^*} (V(., N + s) - V) f_{\Delta_N}(s) ds\right]\end{aligned} \tag{94}$$

which by Assumption 2 is reduced to the following PDE,

$$\begin{aligned}\rho V = {} & \frac{1}{2\alpha}\left(1 - \left(p^c \frac{\partial V}{\partial x}\right)^2\right) + \frac{\partial V}{\partial x}\left[rx + \overline{wh}\left(1 - \frac{1}{\xi}\left(n^* - \frac{\partial V}{\partial x}\int_{I^*}^{N} \frac{e^{\gamma_g j} w_j}{\frac{\partial V}{\partial h_j}} dj\right)\right) - \frac{p^c}{\alpha}\left(1 - p^c \frac{\partial V}{\partial x}\right)\right] \\ & + \frac{1}{2\xi}\int_{I^*}^{N}\left(e^{-\gamma_g j} h_j \frac{\partial V}{\partial h_j} - \overline{wh} w_j e^{\gamma_g j} \frac{\left(\frac{\partial V}{\partial x}\right)^2}{\frac{\partial V}{\partial h_j}}\right) dj.\end{aligned} \tag{95}$$

To solve this problem we employ the following *asatz* for the optimal current-value function

$$V(x, \boldsymbol{h}; I^*, N) = D_0 + D_1 \tilde{Q} + D_2 \tilde{Q}^2 \tag{96}$$

where $\tilde{Q}$ is described by (27) and (28).[24] We have then

$$\frac{\partial V}{\partial x} = D_1 + 2D_2\tilde{Q} \text{ and } \frac{\partial V}{\partial h_j} = \frac{D_j}{\sqrt{2h_j}} \cdot \frac{\partial V}{\partial x} \tag{97}$$

for $j \in [I^*, N]$ where D_j is given by (28). Employing (97), we get

$$\frac{1}{2\alpha}\left(1 - \left(p^c \frac{\partial V}{\partial x}\right)^2\right) = \frac{1}{2\alpha}(1 - D_1^2) + \frac{p^c D_1 D_2}{\alpha}\tilde{Q} + \frac{2(p^c D_2)^2}{\alpha}\tilde{Q}^2 \tag{98}$$

$$rx\frac{\partial V}{\partial x} = rD_1 x + 2rD_2 x\tilde{Q} \tag{99}$$

$$\begin{aligned}\frac{\partial V}{\partial x}\overline{wh}\left(1 - \frac{1}{\xi}\left(n^* - \frac{\partial V}{\partial x}\int_{I^*}^N \frac{e^{\gamma_g j} w_j}{\frac{\partial V}{\partial h_j}} dj\right)\right) &= \overline{wh}(1 - \frac{1}{\xi}n^*)D_1 + 2\overline{wh}(1 - \frac{1}{\xi}n^*)D_2\tilde{Q} \\ &+ \frac{\overline{wh}}{\xi}D_1\int_{I^*}^N \frac{e^{\gamma_g j} w_j}{D_j}\sqrt{2h_j}dj + 2\frac{\overline{wh}}{\xi}D_2\int_{I^*}^N \frac{e^{\gamma_g j} w_j}{D_j}\sqrt{2h_j}dj\tilde{Q}\end{aligned} \tag{100}$$

$$\frac{\partial V}{\partial x}\frac{p^c}{\alpha}\left(1 - p^c\frac{\partial V}{\partial x}\right) = \frac{p^c}{\alpha}D_1 - \frac{(p^c D_1)^2}{\alpha} + 2\left(\frac{p^c}{\alpha}D_2 - \frac{(p^c)^2}{\alpha}D_1 D_2\right)\tilde{Q} - \frac{4\,(p^c D_2)^2}{\alpha}\tilde{Q}^2 \tag{101}$$

$$\frac{1}{2\xi}\int_{I^*}^N e^{-\gamma_g j} h_j \frac{\partial V}{\partial h_j} dj = \frac{D_1}{2\xi}\int_{I^*}^N \frac{e^{-\gamma_g j} D_j}{2}\sqrt{2h_j}dj + \frac{D_2}{\xi}\int_{I^*}^N \frac{e^{-\gamma_g j} D_j}{2}\sqrt{2h_j}dj\tilde{Q} \tag{102}$$

$$\frac{1}{2\xi}\int_{I^*}^N \overline{wh} w_j e^{\gamma_g j}\frac{\left(\frac{\partial V}{\partial x}\right)^2}{\frac{\partial V}{\partial h_j}} dj = \frac{D_1}{2\xi}\int_{I^*}^N \frac{\overline{wh} w_j e^{\gamma_g j}}{D_j}\sqrt{2h_j}dj + \frac{D_2}{\xi}\int_{I^*}^N \frac{\overline{wh} w_j e^{\gamma_g j}}{D_j}\sqrt{2h_j}dj\tilde{Q}. \tag{103}$$

Observe also that

$$\frac{\overline{wh} e^{\gamma_g j} w_j}{2\xi D_j} + \frac{e^{-\gamma j} D_j}{4\xi} = rD_j. \tag{104}$$

Inserting (96) and (98)-(103) into the PDE (95) and employing (104) yields:

$$\begin{aligned}\rho D_0 + \rho D_1\tilde{Q} + \rho D_2\tilde{Q}^2 &= \frac{1}{2\alpha}(1 - D_1^2) + \overline{wh}(1 - \frac{1}{\xi}n^*)D_1 + \frac{p^c}{\alpha}D_1 - \frac{(p^c D_1)^2}{\alpha} \\ &+ \left[rD_1 + \frac{p^c D_1 D_2}{\alpha} + 2\left(\frac{p^c}{\alpha}D_2 - \frac{(p^c)^2}{\alpha}D_1 D_2\right) + 2\overline{wh}(1 - \frac{1}{\xi}n^*)D_2\right]\tilde{Q} \\ &+ \left[2rD_2 + \frac{6(p^c D_2)^2}{\alpha}\right]\tilde{Q}^2.\end{aligned} \tag{105}$$

Thereby we arrive at three equations for the three unknowns D_0, D_1 and D_2. Equating the coefficients of $\tilde{Q}^2$ on left and right-hand sides of (105) yields D_2 as described in (28).[25] Similarly Equating the coefficients of $\tilde{Q}$ on both sides the equality in $\tilde{Q}$ yields D_1 as expressed in (29). One can thereafter derive the expression for D_0 as well, which is given by

$$D_0 = \frac{1}{\rho}\left(\frac{1}{2\alpha}(1 - D_1^2) + \overline{wh}(1 - \frac{1}{\xi}n^*)D_1 + \frac{p^c}{\alpha}D_1 - \frac{(p^c D_1)^2}{\alpha}\right). \tag{106}$$

[24] Observe that the semi-colon in $V(x, \boldsymbol{h}; I^*, N)$ is there to indicate that by Assumption (2) I^* and N are now just constants.

[25] The equality also yields $D_2 = 0$, which results in $D_0 = D_1 = 0$ and hence $V \equiv 0$ the trivial zero solution to the PDE.

□

Proof of Proposition 2.7. As done previously, for ease of notation we suppress the agent and time indexes i and t. The calculations up to the HJB-equation (94) are the same as in the proof of Proposition 2.6, which then by Assumption 2′ is reduced to the following PDE,

$$\rho V = \frac{1}{2\alpha}\left(1-\left(p^c\frac{\partial V}{\partial x}\right)^2\right)+\frac{\partial V}{\partial x}\left[rx+\overline{wh}\left(1-\frac{1}{\xi}\left(n^*-\frac{\partial V}{\partial x}\int_{I^*}^{N}\frac{e^{\gamma_g j}w_j}{\frac{\partial V}{\partial h_j}}dj\right)\right)-\frac{p^c}{\alpha}\left(1-p^c\frac{\partial V}{\partial x}\right)\right]$$
$$+\frac{1}{2\xi}\int_{I^*}^{N}\left(e^{-\gamma_g j}h_j\frac{\partial V}{\partial h_j}-\overline{wh}w_j e^{\gamma_g j}\frac{\left(\frac{\partial V}{\partial x}\right)^2}{\frac{\partial V}{\partial h_j}}\right)dj+\sigma^2 x\left(2D_2\tilde{Q}+D_1\right). \tag{107}$$

To solve this problem we employ the structure of *asatz* as in (96) for the optimal current-value function. We derive then (98) to (103) in the same manner. One difference is that instead of (104) we have the following result:

$$\frac{\overline{wh}e^{\gamma_g j}w_j}{2\xi D_j}+\frac{e^{-\gamma j}D_j}{4\xi}=(r+\sigma^2)D_j. \tag{108}$$

Inserting (96) and (98)-(103) into the PDE (107) and employing (108) yields:

$$\rho D_0+\rho D_1\tilde{Q}+\rho D_2\tilde{Q}^2=\frac{1}{2\alpha}(1-D_1^2)+\overline{wh}(1-\frac{1}{\xi}n^*)D_1+\frac{p^c}{\alpha}D_1-\frac{(p^cD_1)^2}{\alpha}$$
$$+\left[(r+\sigma^2)D_1+\frac{p^cD_1D_2}{\alpha}+2\left(\frac{p^c}{\alpha}D_2-\frac{(p^c)^2}{\alpha}D_1D_2\right)+2\overline{wh}(1-\frac{1}{\xi}n^*)D_2\right]\tilde{Q}$$
$$+\left[2(r+\sigma^2)D_2+\frac{6(p^cD_2)^2}{\alpha}\right]\tilde{Q}^2. \tag{109}$$

Thereby we arrive at three equations for D_0, D_1 and D_2. Equating the coefficients of $\tilde{Q}^2$ and $\tilde{Q}$ on left and right-hand sides of (105) yields the tautology $0=0$, validating the *asatz*.[26] One can thereafter derive the expression for D_0 as well, which is given by (106). □

Proof of Proposition 2.9. The profit function for the producer given (40) and (20) is the following,

$$\pi_{jt}=\begin{cases} p_{jt}\vartheta_M(j)k_{jt}-r_{jt}k_{jt} & \text{for } j\in[N_t-1,I_t^*] \\ p_{jt}\vartheta_L(j)h_{jt}\ell_{jt}-w_{jt}h_{jt}\ell_{jt} & \text{for } j\in[I_t^*,N_t]\end{cases} \tag{110}$$

Perfect competition implies zero profits ($\pi_{jt}=0$) and hence we have,

$$r_{jt}=p_{jt}\vartheta_M(j) \text{ and } w_{jt}=p_{jt}\vartheta_L(j)$$

for $j\in[N_t-1,I_t^*]$ and $j\in[I_t^*,N_t]$ respectively. Utilizing Lemma 2.2 we then get

$$r_{jt}=P_t\left(\frac{Y_t}{k_{jt}}\right)^{\frac{1}{\epsilon}}\vartheta_M^{\frac{\varepsilon-1}{\varepsilon}}(j) \text{ and } w_{jt}=P_t\left(\frac{Y_t}{h_{jt}\ell_{jt}}\right)^{\frac{1}{\epsilon}}\vartheta_L^{\frac{\varepsilon-1}{\varepsilon}}(j), \tag{111}$$

[26] As before, $V\equiv 0$ is the trivial zero solution to the PDE.

for $j \in [N_t - 1, I_t^*]$ and $j \in [I_t^*, N_t]$ respectively. Employing Assumption 4, we realize that

$$\frac{\vartheta_M^{\frac{\varepsilon-1}{\varepsilon}}(j)}{k_{jt}^{\frac{1}{\varepsilon}}} = R_t,$$

where R_t is independent of j. Hence,

$$\frac{\vartheta_M^{\varepsilon-1}(j)}{R_t^{\varepsilon}} = k_{jt}. \tag{112}$$

Integrating both sides over $j \in [N_t - 1, I_t^*]$ - by the capital market clearing condition (41) - and rearranging we get the following expression for R_t

$$R_t = \left(\frac{\int_{N_t-1}^{I_t^*} \vartheta_M^{\varepsilon-1}(j) dj}{X_t} \right)^{\frac{1}{\varepsilon}}. \tag{113}$$

Then (43) and (44) follows from (111), (112) and (113). □

Proof of Lemma 2.3. First consider the condition for agents having manageable human-capital-augmented wage rate without any jump risk $\lambda = 0$

$$w_{jt} h_{ijt} = w_{j't} h_{ij't}$$

we rewrite it as

$$w_{jt} h_{ijt^-} \cdot e^{g_j(a_{ijt}^*)dt} = w_{j't} h_{ij't^-} \cdot e^{g_{j'}(a_{ij't}^*)dt} \tag{114}$$

where

$$t^- = \lim_{\epsilon \downarrow 0, \epsilon > 0} t - \epsilon. \tag{115}$$

With jump risk the condition for agents having manageable human-capital-augmented wage rate

$$\begin{aligned} w_{jt} h_{ijt^-} \cdot e^{g_j(a_{ijt}^*)dt} \cdot 1_{\{h_{ij}(t^-+dt) \neq 0 | h_{ijt^-} \neq 0\}} \\ = w_{j't} h_{ij't^-} \cdot e^{g_{j'}(a_{ij't}^*)dt} \cdot 1_{\{h_{ij'}(t^-+dt) \neq 0 | h_{ij't^-} \neq 0\}} \end{aligned} \tag{116}$$

Observe that

$$1_{\{h_{ij}(t^-+dt) \neq 0 | h_{ijt^-} \neq 0\}} = 1_{\{T_{I_t} > t^-+dt | T_{I_t} > t^-\}} \cdot 1_{\{I_t \Delta_{I_t} < j\}}. \tag{117}$$

where T_{I_t} is the stochastic variable for time of new automation technology arriving. Moreover we have that,

$$\mathbb{E}\left\{1_{\{T_{I_t} > t^-+dt | T_{I_t} > t^-\}}\right\} = \frac{\mathcal{P}\left(T_{I_t} > t^- + dt\right)}{\mathcal{P}\left(T_{I_t} > t^-\right)} = \frac{e^{\lambda(t^-+dt)}}{e^{\lambda t^-}} = e^{\lambda dt} \tag{118}$$

where we have used that the distribution of a arrival time of Poisson processes $Poi(\lambda(t))$ are given by Exponential distributions $Exp(\lambda(t))$. Furthermore we have that,

$$\mathbb{E}\left\{1_{\{I_t+\Delta_{I_t}<j\}}\right\} = F_{I_t+\Delta_{I_t}}(j) \tag{119}$$

assuming continuity of density $f_{\Delta_{I_t}}(j)$ over the support $j \in [N_t, I_t^*]$. Thus by (117)-(119), taking expectation on both sides of (116) and comparing the result to (114) , the condition for agents having manageable human-capital-augmented wage rate becomes (52). Finally, (53) follows from correspondingly similar calculations as above. □

Proof of Proposition 2.10. As done previously, for ease of notation we suppress the agent and time indexes i and t. The calculations up to the HJB-equation (94) are the same as in the proof of Proposition 2.6, which then by Assumption 2″ and Lemma 2.3 is reduced to the following PIDE,

$$\begin{aligned}\rho V = &\frac{1}{2\alpha}\left(1-\left(p^c\frac{\partial V}{\partial x}\right)^2\right) + \frac{\partial V}{\partial x}\left[rx + \overline{wh}\left(1-\frac{1}{\xi}\left(n^* - \frac{\partial V}{\partial x}\int_{I^*}^N \frac{e^{\gamma_g j} w_j F_{\Delta_{I_t}}(j)}{\frac{\partial V}{\partial h_j}}dj\right)\right) - \frac{p^c}{\alpha}\left(1-p^c\frac{\partial V}{\partial x}\right)\right] \\ &+ \frac{1}{2\xi}\int_{I^*}^N \left(e^{-\gamma_g j}h_j\frac{\partial V}{\partial h_j} - \overline{wh}w_j F_{\Delta_{I_t}}(j)e^{\gamma_g j}\frac{\left(\frac{\partial V}{\partial x}\right)^2}{\frac{\partial V}{\partial h_j}}\right)dj + \sigma^2 x\left(2D_2\tilde{Q} + D_1\right) \\ &+ \lambda\left[\int_0^{1-n_t^*}(V(.,N+s)-V)f_{\Delta_{N_t}}(s)ds\right].\end{aligned} \tag{120}$$

To solve this problem we employ the structure of *asatz* as in (96) for the optimal current-value function but D_{ijt}, D_{1t} and D_{2t} are instead given by (57), (58) and (59). Employing (97), we get

$$\frac{1}{2\alpha}\left(1-\left(p^c\frac{\partial V}{\partial x}\right)^2\right) = \frac{1}{2\alpha}(1-D_1^2) + \frac{p^c D_1 D_2}{\alpha}\tilde{Q} + \frac{2(p^c D_2)^2}{\alpha}\tilde{Q}^2 \tag{121}$$

$$rx\frac{\partial V}{\partial x} = rD_1 x + 2rD_2 x\tilde{Q} \tag{122}$$

$$\begin{aligned}\frac{\partial V}{\partial x}\overline{wh}\left(1-\frac{1}{\xi}\left(n^* - \frac{\partial V}{\partial x}\int_{I^*}^N \frac{e^{\gamma_g j} w_j F_{\Delta_I}(j)}{\frac{\partial V}{\partial h_j}}dj\right)\right) &= \overline{wh}(1-\frac{1}{\xi}n^*)D_1 + 2\overline{wh}(1-\frac{1}{\xi}n^*)D_2\tilde{Q} \\ &+ \frac{\overline{wh}}{\xi}D_1\int_{I^*}^N \frac{e^{\gamma_g j}w_j}{D_j}\sqrt{2h_j}dj + 2\frac{\overline{wh}}{\xi}D_2\int_{I^*}^N \frac{e^{\gamma_g j}w_j F_{\Delta_I}(j)}{D_j}\sqrt{2h_j}dj\tilde{Q}\end{aligned} \tag{123}$$

$$\frac{\partial V}{\partial x}\frac{p^c}{\alpha}\left(1-p^c\frac{\partial V}{\partial x}\right) = \frac{p^c}{\alpha}D_1 - \frac{(p^c D_1)^2}{\alpha} + 2\left(\frac{p^c}{\alpha}D_2 - \frac{(p^c)^2}{\alpha}D_1 D_2\right)\tilde{Q} - \frac{4\,(p^c D_2)^2}{\alpha}\tilde{Q}^2 \tag{124}$$

$$\frac{1}{2\xi}\int_{I^*}^N e^{-\gamma_g j}h_j\frac{\partial V}{\partial h_j}dj = \frac{D_1}{2\xi}\int_{I^*}^N \frac{e^{-\gamma_g j}D_j}{2}\sqrt{2h_j}dj + \frac{D_2}{\xi}\int_{I^*}^N \frac{e^{-\gamma_g j}D_j}{2}\sqrt{2h_j}dj\tilde{Q} \tag{125}$$

$$\frac{1}{2\xi}\int_{I^*}^N \overline{wh}w_j F_{\Delta_{I_t}}(j)e^{\gamma_g j}\frac{\left(\frac{\partial V}{\partial x}\right)^2}{\frac{\partial V}{\partial h_j}}dj = \frac{D_1}{2\xi}\int_{I^*}^N \frac{\overline{wh}w_j F_{\Delta_{I_t}}(j)e^{\gamma_g j}}{D_j}\sqrt{2h_j}dj \tag{126}$$

$$+ \frac{D_2}{\xi}\int_{I^*}^N \frac{\overline{wh}w_j F_{\Delta_{I_t}}(j)e^{\gamma_g j}}{D_j}\sqrt{2h_j}dj\tilde{Q}. \tag{127}$$

Observe also that

$$\frac{\overline{wh}e^{\gamma_g j}w_j F_{\Delta_{I_t}}(j)}{2\xi D_j} + \frac{e^{-\gamma j}D_j}{4\xi} = (r+\sigma^2-\lambda\mathcal{P}_{\Delta_{N_t}})D_j. \tag{128}$$

Inserting (96) and (121)-(127) into the PDE (120) and employing (128) and Lemma (2.4) yields:

$$\begin{aligned}\rho D_0 + \rho D_1 \tilde{Q} + \rho D_2 \tilde{Q}^2 = \frac{1}{2\alpha}(1 - D_1^2) + \overline{wh}(1 - \frac{1}{\xi} n^*) D_1 + \frac{p^c}{\alpha} D_1 - \frac{(p^c D_1)^2}{\alpha} \\ + \left[(r + \sigma^2 - \lambda \mathcal{P}_{\Delta_{N_t}}) D_1 + \frac{p^c D_1 D_2}{\alpha} + 2 \left(\frac{p^c}{\alpha} D_2 - \frac{(p^c)^2}{\alpha} D_1 D_2 \right) + 2\overline{wh}(1 - \frac{1}{\xi} n^*) D_2 \right] \tilde{Q} \\ + \left[2(r + \sigma^2 - \lambda \mathcal{P}_{\Delta_{N_t}}) D_2 + \frac{6(p^c D_2)^2}{\alpha} \right] \tilde{Q}^2. \end{aligned} \tag{129}$$

Observe that we have assumed that agents are acting slightly more conservative by inserting the lower bound given in Lemma 2.4. Thereby we arrive at three equations for D_0, D_1 and D_2. Equating the coefficients of $\tilde{Q}^2$ and $\tilde{Q}$ on left and right-hand sides of (129) yields the tautology $0 = 0$, validating the *asatz*.[27] One can thereafter derive the expression for D_0 as well, which is given by (106). □

[27] As before, $V \equiv 0$ is the trivial zero solution to the PIDE.

Appendix B - Figures

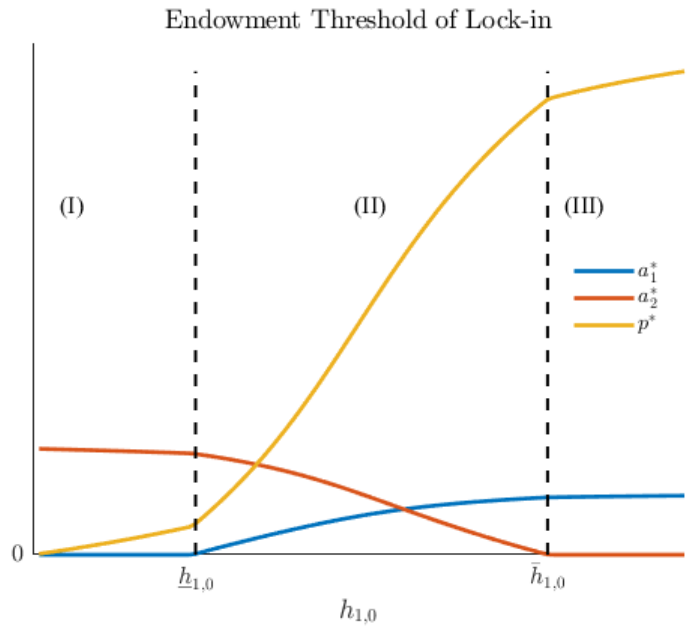

Figure 1: Example with Diversifiers

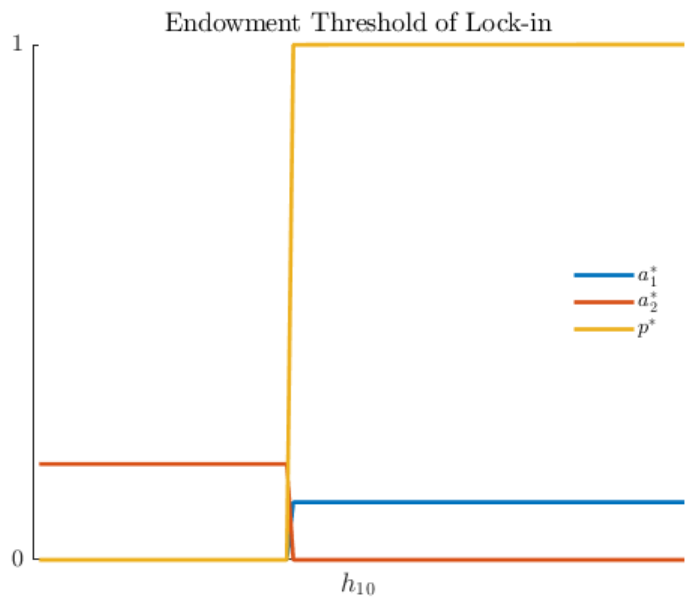

Figure 2: Example with no Diversifiers

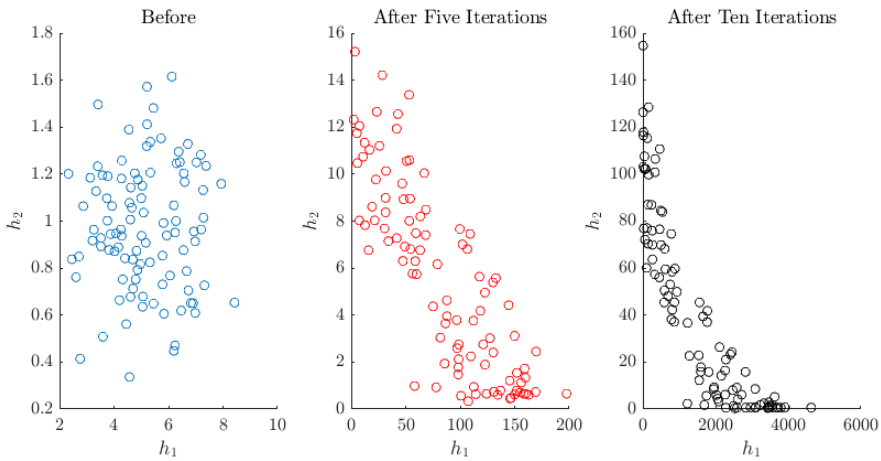

Figure 3: Example with Diversifiers

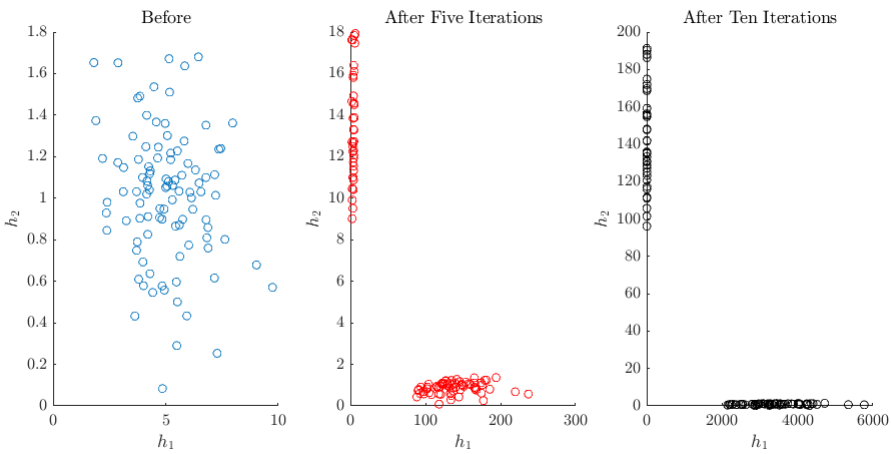

Figure 4: Example with no Diversifiers